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MATHEMATICS

TARGET IIT JEE

INVERSE TRIGONOMETRIC FUNCTIONS

THEORY AND EXERCISE BOOKLET

CONTENTS

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JEE Syllabus :

Inverse trigonometric functions (principal value only)

A. INTRODUCTION

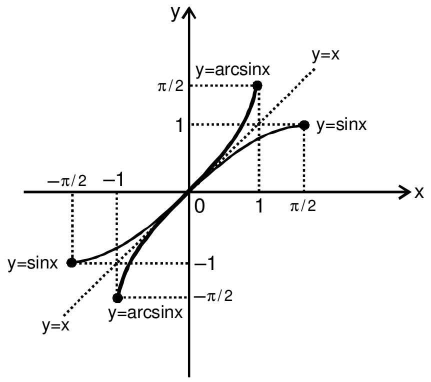
$\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$ etc. represents angles or numbers whose values of sine, cosine and tangent is 'x', provided that the value in numerical form is smallest. These can be written as arc sin x, arc cos x etc. If two angles whose modulus is equal, in which one is positive and other is negative then we take positive sign.

B. DOMAIN & PRINCIPLE VALUE RANGE OF INVERSE TRIGONOMETRIC FUNCTIONS

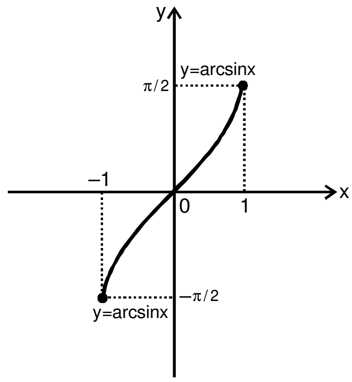
S.No.	Function	Domain	Principle value range (PVR)
1.	$y = \sin^{-1} x$	$x \in [-1, 1]$	$y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
2.	$y = \cos^{-1} x$	$x \in [-1, 1]$	$y \in [0, \pi]$
3.	$y = \tan^{-1} x$	$x \in R$	$y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
4.	$y = \cot^{-1} x$	$x \in R$	$y \in (0, \pi)$
5.	$y = \sec^{-1} x$	$x \in (-\infty, -1] \cup [1, \infty)$	$y \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$
6.	$y = \text{cosec}^{-1} x$	$x \in (-\infty, -1] \cup [1, \infty)$	$y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

C. GRAPH OF INVERSE TRIGONOMETRIC FUNCTIONS

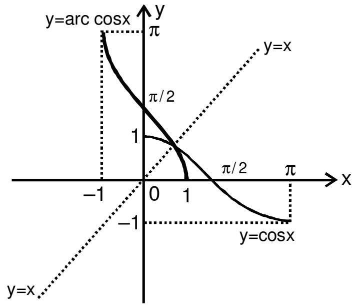
(a) $f : [-\pi/2, \pi/2] \to [-1, 1]$
$f(x) = \sin x$



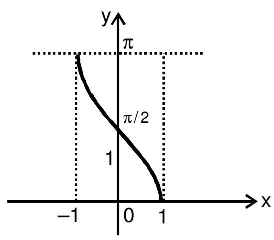
$f^{-1} : [-1, 1] \to [-\pi/2, \pi/2]$
$f^{-1}(x) = \sin^{-1}x$



(b) $f : [0, \pi] \to [-1, 1]$
$f(x) = \cos x$

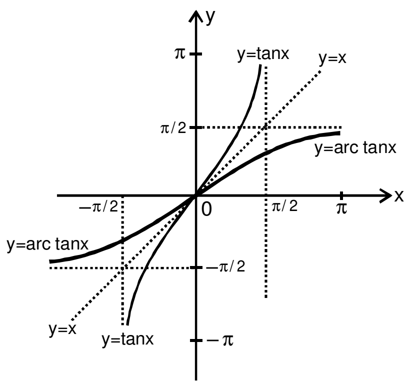


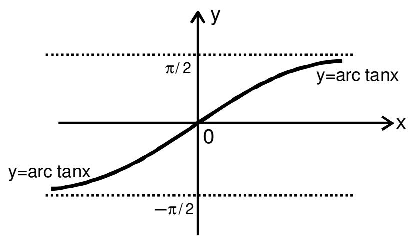
$f^{-1} : [-1, 1] \to [0, \pi]$
$f^{-1}(x) = \cos^{-1} x$



(c) $f : (-\pi/2, \pi/2) \to R$ $f^{-1} : R \to (-\pi/2, \pi/2)$

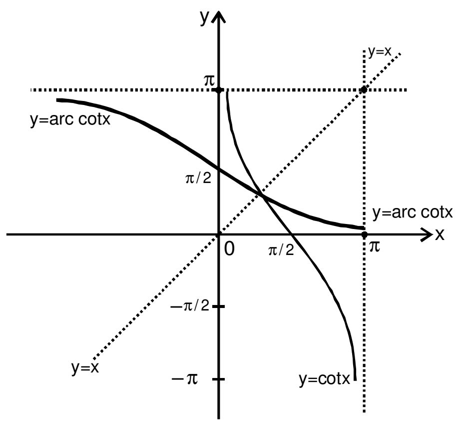
$f(x) = \tan x$ $f^{-1}(x) = \tan^{-1} x$

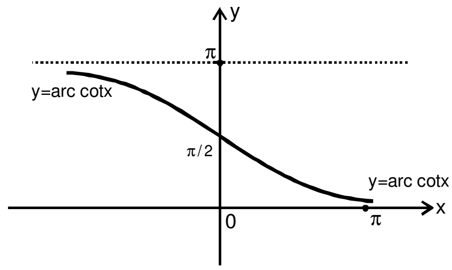




(d) $f : (0, \pi) \to R$ $f^{-1} : R \to (0, \pi)$

$f(x) = \cot x$ $f^{-1}(x) = \cot^{-1} x$



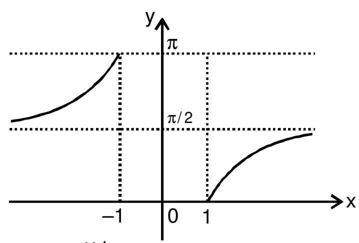


(e) $f : [0, \pi/2) \cup (\pi/2, \pi] \to (-\infty, -1] \cup [1, \infty)$

$f(x) = \sec x$

$f^{-1} : (-\infty, -1] \cup [1, \infty) \to [0, \pi/2) \cup [\pi/2, \pi]$

$f^{-1}(x) = \sec^{-1} x$

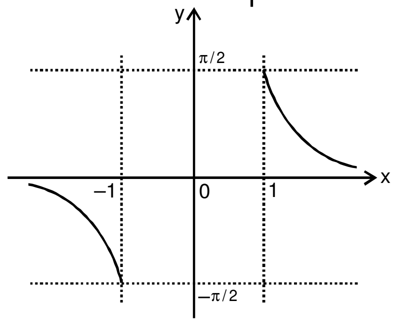


(f) $f : [-\pi/2, 0) \cup (0, \pi/2] \to (-\infty, -1] \cup [1, \infty)$

$f(x) = \text{cosec } x$

$f^{-1} : (-\infty, -1] \cup [1, \infty) \to [-\pi/2, 0) \cup (0, \pi/2]$

$f^{-1}(x) = \text{cosec}^{-1} x$



Basis on the above discussion we get following results :

(i) All inverse trigonometric functions shows angle.

(ii) If $x \geq 0$ then all six trigonometric functions $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, $\sec^{-1} x$, $\text{cosec}^{-1} x$, $\cot^{-1} x$ shows acute angle.

(iii) If $x < 0$ then $\sin^{-1} x$, $\tan^{-1} x$ and $\text{cosec}^{-1} x$, shows angle between $-\pi/2$ to 0 (IV quadrant)

(iv) If $x < 0$ then $\cos^{-1} x$, $\cot^{-1} x$ and $\sec^{-1} x$ shows obtuse angle (II quadrant)

(v) III quadrant never used in inverse trigonometric functions.

Ex.1 Find the domain of the following functions.

(i) $\sin^{-1} \ell n\, x$ **(ii)** $\cos^{-1} [x]$ **(iii)** $\sin^{-1} \{x\}$

Sol. **(i)** $f(x) = \sin^{-1} \ell n\, x \Rightarrow -1 \le \ell n\, x \le 1 \Rightarrow \frac{1}{e} \le x \le e$

(ii) $f(x) = \cos^{-1} [x] \Rightarrow -1 \le [x] \le 1 \Rightarrow [x] = -1, 0, 1 \Rightarrow x \in [-1, 0) \cup [0, 1) \cup [1, 2) \Rightarrow x \in [-1, 2)$

(iii) $f(x) = \sin^{-1} \{x\} \Rightarrow -1 \le \{x\} \le 1 \Rightarrow x \in R$

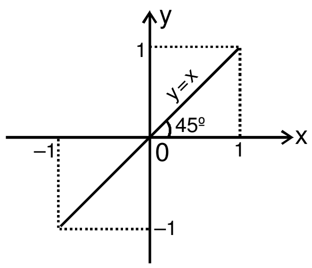
Ex.2 $\tan^{-1} (1) + \cos^{-1} \left(-\frac{1}{2}\right) + \sin^{-1} \left(-\frac{1}{2}\right)$ is equal to

Sol. $\tan^{-1} (1) + \cos^{-1} \left(-\frac{1}{2}\right) + \sin^{-1} \left(-\frac{1}{2}\right) = \frac{\pi}{4} + \frac{2\pi}{3} - \frac{\pi}{6} = \frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{4}$

D. PROPERTIES OF INVERSE TRIGONOMETRIC FUNCTIONS

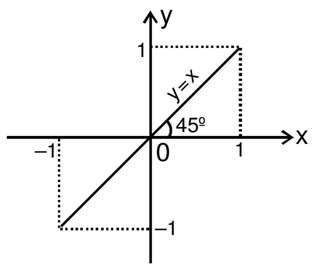
P-1 **(i)** $y = \sin (\sin^{-1} x) = x$

$x \in [-1, 1], y \in [-1, 1]$



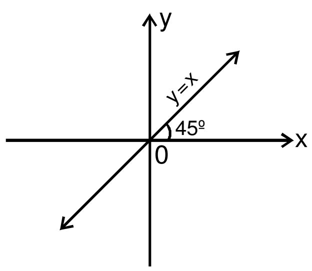
(ii) $y = \cos (\cos^{-1} x) = x$

$x \in [-1, 1], y \in [-1, 1]$



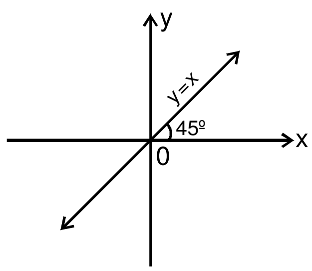
(iii) $y = \tan (\tan^{-1} x) = x$

$x \in R, y \in R$



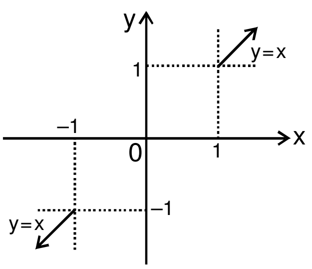
(iv) $y = \cot (\cot^{-1} x) = x$

$x \in R, y \in R$



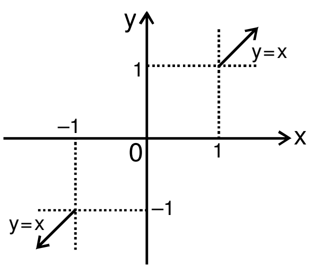
(v) $y = \text{cosec} (\text{cosec}^{-1} x) = x$

$|x| \ge 1, |y| \ge 1$



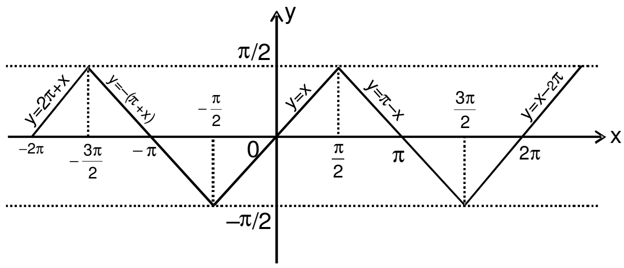
(vi) $y = \sec (\sec^{-1} x) = x$

$|x| \ge 1 ; |y| \ge 1$



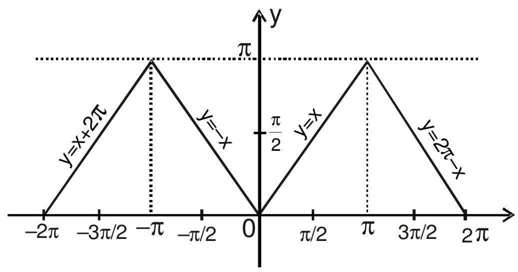
(vii) $y = \sin^{-1}(\sin x)$, $x \in R$, $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, is periodic function with period 2π and it is an odd function

$$\sin^{-1}(\sin x) = \begin{cases} -\pi - x \ , & -\pi \le x \le -\frac{\pi}{2} \\ x \ , & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ \pi - x \ , & \frac{\pi}{2} \le x \le \pi \end{cases}$$



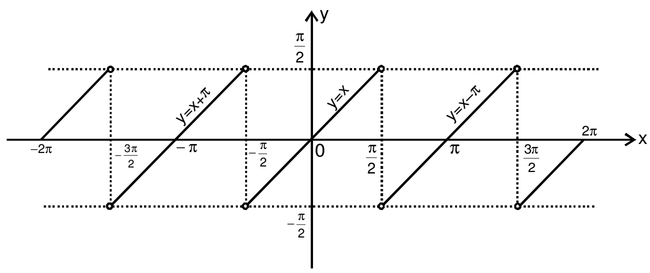
(viii) $y = \cos^{-1}(\cos x)$, $x \in R$, $y \in [0, \pi]$, is periodic function with period 2π and it is an even function

$$\cos^{-1}(\cos x) = \begin{cases} -x \ , & -\pi \le x \le 0 \\ x \ , & 0 < x \le \pi \\ 2\pi - x \ , & \pi < x \le 2\pi \\ x - 2\pi \ , & 2\pi < x \le 3\pi \end{cases}$$



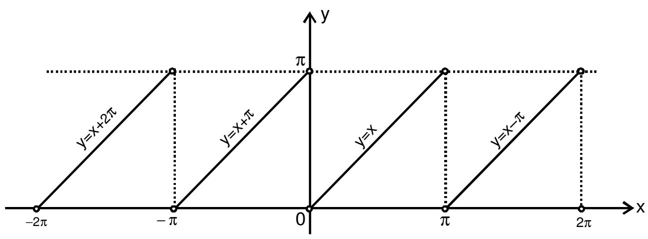
(ix) $y = \tan^{-1}(\tan x)$, $x \in R - \left\{(2n+1)\frac{\pi}{2}, n \in I\right\}$; $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is periodic function with period π and it is an odd function

$$\tan^{-1}(\tan x) = \begin{cases} x + \pi & ; -\frac{3\pi}{2} < x < -\frac{\pi}{2} \\ x & ; -\frac{\pi}{2} < x < \frac{\pi}{2} \\ x - \pi & ; \frac{\pi}{2} < x < \frac{3\pi}{2} \end{cases}$$

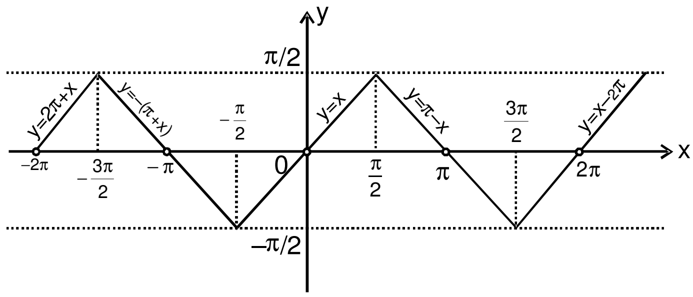


(x) $y = \cot^{-1}(\cot x)$, $x \in R - \{n\pi, n \in I\}$, $y \in [0, \pi]$, is periodic function with period π and it is neither an even nor odd function

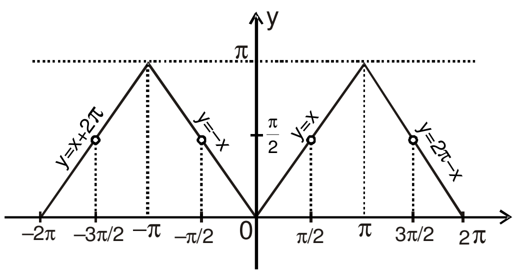
$$\cot^{-1}(\cot x) = \begin{cases} x + \pi \ ; & -\pi < x < 0 \\ x \ ; & 0 < x < \pi \\ x - \pi \ ; & \pi < x < 2\pi \end{cases}$$



(xi) $y = \text{cosec}^{-1}(\text{cosec } x)$, $x \in R - \{n\pi, n \in I\}$, $y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ is periodic function with period 2π and it is an odd function



(xii) $y = \sec^{-1}(\sec x)$, $x \in R - \left\{(2n-1)\frac{\pi}{2}, n \in I\right\}$, $y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$ is periodic function with period 2π and it is an even function



Ex.3 Evaluate following

(i) $\sin(\cos^{-1} 3/5)$ **(ii)** $\cos(\tan^{-1} 3/4)$ **(iii)** $\sin\left(\frac{\pi}{2} - \sin^{-1}\left(-\frac{1}{2}\right)\right)$

Sol. **(i)** Let $\cos^{-1} 3/5 = \theta$ then $\cos\theta = 3/5 \Rightarrow \sin\theta = 4/5 \quad \therefore \sin(\cos^{-1} 3/5) = \sin\theta = 4/5$

(ii) Let $\tan^{-1} 3/4 = \theta$ then $\tan\theta = 3/4 \Rightarrow \cos\theta = 4/5 \quad \therefore \cos(\tan^{-1} 3/4) = \cos\theta = 4/5$

(iii) $\sin\left(\frac{\pi}{2} - \sin^{-1}\left(-\frac{1}{2}\right)\right) = \sin\left(\frac{\pi}{2} - \left(-\frac{\pi}{6}\right)\right) = \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$

Ex.4 Define the function, $f(x) = \cos^{-1}(\cos x) - \sin^{-1}(\sin x)$ in $[0, 2\pi]$ and find the area bounded by the graph of the function and the x – axis.

Sol. $\cos^{-1}(\cos x) = \begin{cases} x & 0 \le x \le \pi \\ 2\pi - x & \pi \le x \le 2\pi \end{cases}$; $\sin^{-1}(\sin x) = \begin{cases} x & 0 \le x \le \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} \le x \le \frac{3\pi}{2} \\ x - 2\pi & \frac{3\pi}{2} \le x \le 2\pi \end{cases}$

Hence $f(x) = \begin{cases} 0 & \text{if } x \in \left[0, \frac{\pi}{2}\right] \\ 2x - \pi & \text{if } x \in \left[\frac{\pi}{2}, \pi\right] \\ \pi & \text{if } x \in \left[\pi, \frac{3\pi}{2}\right] \\ 4\pi - 2x & \text{if } x \in \left[\frac{3\pi}{2}, 2\pi\right] \end{cases}$ Area $= \left(\frac{3\pi}{2} + \frac{\pi}{2}\right) - \frac{\pi}{2} = \pi^2$

Ex.5 Evaluate each of the following

(i) $\sin\left(\frac{\pi}{3}-\sin^{-1}\frac{4}{5}\right)$, **(ii)** $\cos\left(\sin^{-1}\frac{4}{5}+\cos^{-1}\frac{2}{3}\right)$, **(iii)** $\sin(\tan^{-1}3+\tan^{-1}2)$

Sol. **(i)** $\sin\left(\frac{\pi}{3}-\sin^{-1}\frac{4}{5}\right) = \sin\frac{\pi}{3}\cos\left(\sin^{-1}\frac{4}{5}\right) - \cos\frac{\pi}{3}\sin\left(\sin^{-1}\frac{4}{5}\right)$

$=\frac{\sqrt{3}}{2}\sqrt{1-(4/5)^2}-\frac{1}{2}.\frac{4}{5}=\frac{\sqrt{3}}{2}.\frac{3}{5}-\frac{4}{10}=\frac{3\sqrt{3}-4}{10}$.

(ii) $\cos\left(\sin^{-1}\frac{4}{5}+\cos^{-1}\frac{2}{3}\right) = \cos\left(\sin^{-1}\frac{4}{5}\right)\cos\left(\cos^{-1}\frac{2}{3}\right) - \sin\left(\sin^{-1}\frac{4}{5}\right)\sin\left(\cos^{-1}\frac{2}{3}\right)$

$=\sqrt{1-(4/5)^2}.\frac{2}{3}-\frac{4}{5}.\sqrt{1-(2/3)^2}=\frac{3}{5}.\frac{2}{3}-\frac{4}{5}.\frac{\sqrt{5}}{3}=\frac{6-4\sqrt{5}}{15}$.

(iii) $\sin(\tan^{-1}3+\tan^{-1}2) = \sin(\tan^{-1}3)\cos(\tan^{-1}2)+\cos(\tan^{-1}3)\sin(\tan^{-1}2)$

$=\frac{3}{\sqrt{1+3^2}}.\frac{1}{\sqrt{1+2^2}}+\frac{1}{\sqrt{1+3^2}}.\frac{2}{\sqrt{1+2^2}}=\frac{3}{\sqrt{10}\sqrt{5}}=\frac{1}{\sqrt{2}}$

Ex.6 Let $y = \sin^{-1}(\sin 8) - \tan^{-1}(\tan 10) + \cos^{-1}(\cos 12) - \sec^{-1}(\sec 9) + \cot^{-1}(\cot 6) - \text{cosec}^{-1}(\text{cosec } 7)$.
If y simplifies to $a\pi + b$ then find $(a - b)$.

Sol. $\sin^{-1}(\sin 8) = \sin^{-1}(\sin(3\pi-8)) = 3\pi - 8$

$\tan^{-1}(\tan 10) = \tan^{-1}(\tan(10-3\pi)) = 10 - 3\pi$

$\cos^{-1}(\cos 12) = \cos^{-1}(\cos(4\pi-12)) = 4\pi - 12$

$\sec^{-1}(\sec 9) = \sec^{-1}(\sec(9-2\pi)) = 9 - 2\pi$

$\cot^{-1}(\cot 6) = \cot^{-1}(\cot(6-\pi)) = 6 - \pi$

$\text{cosec}^{-1}(\text{cosec } 7) = \text{cosec}^{-1}(\text{cosec}(7-2\pi)) = 7 - 2\pi$

$y = (3\pi - 8) + (3\pi - 10) + (4\pi - 12) + (2\pi - 9) + (-\pi + 6) + (2\pi - 7) = 13\pi - 40$

$\Rightarrow a = 13$ and $b = -40$ $\quad\Rightarrow a - b = 13 - (-40) = 53$

P-2 **(i)** $\text{cosec}^{-1}x = \sin^{-1}\frac{1}{x}$; $|x| \ge 1$ **(ii)** $\sec^{-1}x = \cos^{-1}\frac{1}{x}$; $|x| \ge 1$

(iii) $\cot^{-1}x = \begin{cases}\tan^{-1}\frac{1}{x}, & x > 0\\ \pi+\tan^{-1}\frac{1}{x}, & x < 0\end{cases}$

P-3 **(i)** $\sin^{-1}(-x) = -\sin^{-1}x$; $-1 \le x \le 1$ **(ii)** $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x$; $|x| \ge 1$
(iii) $\tan^{-1}(-x) = -\tan^{-1}x$; $x \in R$ **(iv)** $\cot^{-1}(-x) = \pi - \cot^{-1}x$; $x \in R$
(v) $\cos^{-1}(-x) = \pi - \cos^{-1}x$; $-1 \le x \le 1$ **(vi)** $\sec^{-1}(-x) = \pi - \sec^{-1}x$; $|x| \ge 1$

P-4 **(i)** $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$; $|x| \le 1$ **(ii)** $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$; $x \in R$

(iii) $\sec^{-1}x + \text{cosec}^{-1}x = \frac{\pi}{2}$; $|x| \ge 1$

Ex.7 Find the value of $\sin^{-1}(-\sqrt{3}/2) + \cos^{-1}(\cos(7\pi/6))$.

Sol. $\sin^{-1}(-\sqrt{3}/2) = -\sin^{-1}(\sqrt{3}/2) = -\pi/3$ and $\cos^{-1}(\cos(7\pi/6)) = \cos^{-1}\cos(2\pi - 5\pi/6) = \cos^{-1}\cos(5\pi/6) = 5\pi/6$

hence $\sin^{-1}(-\sqrt{3}/2) + \cos^{-1}(\cos(7\pi/6)) = -\dfrac{\pi}{3} + \dfrac{5\pi}{6} = \dfrac{\pi}{2}$

Ex.8 Find the value of **(i)** $\sin^{-1}(\sin \pi/4)$ **(ii)** $\cos^{-1}\left(\cos\dfrac{7\pi}{6}\right)$

Sol. **(i)** $\sin^{-1}(\sin \pi/4) = \dfrac{\pi}{4}$

(ii) $\cos^{-1}\left(\cos\dfrac{7\pi}{6}\right) \neq \dfrac{7\pi}{6}$, because $\dfrac{7\pi}{6}$, does not lies between 0 and π

Now $\cos^{-1}\left(\cos\dfrac{7\pi}{6}\right) = \cos^{-1}\left(\cos\left(2\pi - \dfrac{5\pi}{6}\right)\right) = \cos^{-1}\left(\cos\dfrac{5\pi}{6}\right) = \dfrac{5\pi}{6}$ $\left\{\because \dfrac{7\pi}{6} = 2\pi - \dfrac{5\pi}{6}\right\}$

P-5 **(i)**
$$\tan^{-1}x + \tan^{-1}y = \begin{cases} \tan^{-1}\dfrac{x+y}{1-xy} & ;\ \text{if } x>0, y>0 \text{ and } xy<1 \\ \pi + \tan^{-1}\dfrac{x+y}{1-xy} & ;\ \text{if } x>0, y>0 \text{ and } xy>1 \\ \dfrac{\pi}{2} & ;\ \text{if } x>0, y>0 \text{ and } xy=1 \end{cases}$$

(ii) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\dfrac{x-y}{1+xy}$ if $x > 0,\ y > 0$

(iii) $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\left[\dfrac{x+y+z-xyz}{1-xy-yz-zx}\right]$ If $x > 0,\ y > 0,\ z > 0$ and $xy + yz + zx < 1$

P-6 **(i)**
$$\sin^{-1}x + \sin^{-1}y = \begin{cases} \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right] & ;\ \text{if } x\ge 0, y\ge 0 \text{ and } (x^2+y^2)\le 1 \\ \pi - \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right] & ;\ \text{if } x\ge 0, y\ge 0 \text{ and } (x^2+y^2)>1 \end{cases}$$

(ii) $\sin^{-1}x - \sin^{-1}y = \sin^{-1}\left[x\sqrt{1-y^2} - y\sqrt{1-x^2}\right]$; If $x \ge 0,\ y \ge 0$

(iii) $\cos^{-1}x + \cos^{-1}y = \cos^{-1}\left[xy - \sqrt{1-x^2}\sqrt{1-y^2}\right]$; If $x \ge 0,\ y \ge 0$

(iv)
$$\cos^{-1}x - \cos^{-1}y = \begin{cases} \cos^{-1}(xy + \sqrt{1-x^2}\ \sqrt{1-y^2}) & ;\ x<y,\ x,y>0 \\ -\cos^{-1}(xy + \sqrt{1-x^2}\ \sqrt{1-y^2}) & ;\ x>y,\ x,y>0 \end{cases}$$

Ex.9 Prove that, $\sin^{-1}\frac{1}{3} + \sin^{-1}\frac{1}{3\sqrt{11}} + \sin^{-1}\frac{3}{\sqrt{11}} = \frac{\pi}{2}$.

Sol. $\tan^{-1}\frac{1}{2\sqrt{2}} + \tan^{-1}\frac{1}{7\sqrt{2}} + \tan^{-1}\frac{3}{\sqrt{2}} = \tan^{-1}\frac{\frac{1}{2\sqrt{2}}+\frac{1}{7\sqrt{2}}}{1-\frac{1}{28}} + \tan^{-1}\frac{3}{\sqrt{2}}$

$= \tan^{-1}\frac{9\sqrt{2}}{27} + \tan^{-1}\frac{3}{\sqrt{2}} = \tan^{-1}\frac{\sqrt{2}}{3} + \tan^{-1}\frac{3}{\sqrt{2}} = \cot^{-1}\frac{3}{\sqrt{2}} + \tan^{-1}\frac{3}{\sqrt{2}} = \frac{\pi}{2}$

Ex.10 Prove that $\sin^{-1} 4/5 + \sin^{-1} 5/13 + \sin^{-1} 16/65 = \pi/2$

Sol. L.H.S. $= \sin^{-1}\left(\frac{4}{5}.\sqrt{1-\frac{25}{169}}+\sqrt{1-\frac{16}{25}}.\frac{5}{13}\right) + \sin^{-1}\frac{16}{65} = \sin^{-1}\frac{63}{65} + \sin^{-1}\frac{16}{65}$

$= \sin^{-1}\left(\frac{63}{65}.\sqrt{1-\left(\frac{16}{65}\right)^2}+\frac{16}{65}.\sqrt{1-\left(\frac{63}{65}\right)^2}\right) = \sin^{-1}\left(\frac{63^2+16^2}{65^2}\right) = \sin^{-1} 1 = \frac{\pi}{2}$ = R.H.S.

Ex.11 Prove that $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}$

Sol. $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8} = \left(\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{5}\right) + \tan^{-1}\frac{1}{8}$

$= \tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{5}}{1-\frac{1}{2}\times\frac{1}{5}}\right) + \tan^{-1}\left(\frac{1}{8}\right)$ {using; $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$, if $xy < 1$}

$= \tan^{-1}\left(\frac{7}{9}\right) + \tan^{-1}\left(\frac{1}{8}\right) = \tan^{-1}\left(\frac{\frac{7}{9}+\frac{1}{8}}{1-\frac{7}{9}\times\frac{1}{8}}\right) = \tan^{-1}\left(\frac{65}{65}\right) = \tan^{-1}(1) = \frac{\pi}{4}$

Ex.12 Find the value of $\sin^{-1}\frac{3}{\sqrt{73}} + \cos^{-1}\frac{11}{\sqrt{146}} + \cot^{-1}\sqrt{3}$.

Sol. $\tan^{-1}\frac{3}{8} + \tan^{-1}\frac{5}{11} + \cot^{-1}\sqrt{3} = \tan^{-1}\left[\frac{\frac{3}{8}+\frac{5}{11}}{1-\frac{3}{8}.\frac{5}{11}}\right] + \cot^{-1}\sqrt{3} = \tan^{-1}(1) + \frac{\pi}{6} = \frac{\pi}{4} + \frac{\pi}{6} = \frac{5\pi}{12}$

Ex.13 Show that $\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{15}{17} = \pi - \sin^{-1}\frac{84}{85}$

Sol. Since $\frac{3}{5} > 0$, $\frac{15}{17} > 0$ and $\left(\frac{3}{5}\right)^2+\left(\frac{15}{17}\right)^2 = \frac{8226}{7225} > 1$

$\therefore \sin^{-1}\frac{3}{5} + \sin^{-1}\frac{15}{17} = \pi - \sin^{-1}\left(\frac{3}{5}\sqrt{1-\frac{225}{289}}+\frac{15}{17}\sqrt{1-\frac{9}{25}}\right) = \pi - \sin^{-1}\left(\frac{3}{5}.\frac{8}{17}+\frac{15}{17}.\frac{4}{5}\right) = \pi - \sin^{-1}\frac{84}{85}$

Ex.14 If $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \frac{3\pi}{2}$ then prove that xy + yz + zx = 1

Sol. Since $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \frac{3\pi}{2}$ $\Rightarrow$ $\tan^{-1} x + \tan^{-1} y = \frac{3\pi}{2} - \tan^{-1} z$

$\Rightarrow$ $\tan(\tan^{-1} x + \tan^{-1} y) = \tan\left(\frac{3\pi}{2} - \tan^{-1} z\right)$ $\Rightarrow$ $\frac{x+y}{1-xy} = \cot(\tan^{-1} z)$ ...(1)

Case (I) : If z > 0 then $\tan^{-1} z = \cot^{-1}\left(\frac{1}{z}\right)$ $\Rightarrow$ $\cot(\tan^{-1} z) = \cot\left(-\pi + \cot^{-1}\frac{1}{z}\right) = \frac{1}{z}$ ...(2)

Case (II) : If z < 0 then $\tan^{-1} z = -\pi + \cot^{-1}\left(\frac{1}{z}\right)$

$\Rightarrow \cot(\tan^{-1} z) = \cot\left(-\pi + \cot^{-1}\frac{1}{z}\right) = \cot\left(\cot^{-1}\frac{1}{z}\right) = \frac{1}{z}$ ...(3)

From (1), (2) and (3) we get $\frac{x+y}{1-xy} = \frac{1}{z}$ or zx + yz = 1 – xy or xy + yz + zx = 1.

Ex.15 If $\cos^{-1} x/2 + \cos^{-1} y/3 = \theta$, prove that $9x^2 + 12\,xy\cos\theta + 4y^2 = 36\sin^2\theta$

Sol. Let $\cos^{-1} x/2 = \alpha$, and $\cos^{-1} y/3 = \beta$ $\therefore$ $\cos\alpha = x/2$ and $\cos\beta = y/3$.
Given, $\alpha+\beta=\theta$ $\therefore$ $\cos(\alpha+\beta) = \cos\theta$

or $\cos\alpha\cos\beta - \sin\alpha\sin\beta = \theta$ or $\frac{x}{2}.\frac{y}{3} - \sqrt{1-\frac{x^2}{4}}\sqrt{1-\frac{y^2}{9}} = \cos\theta$

or $\frac{xy}{6} - \frac{\sqrt{4-x^2}.\sqrt{9-y^2}}{6} = \cos\theta$ or $(xy - 6\cos\theta)^2 = (4 - x^2)(9 - y^2)$

or $x^2y^2 + 36\cos^2\theta - 12xy\cos\theta = 36 - 9x^2 - 4y^2 + x^2y^2$
or $9x^2 - 12y\cos\theta + 4y^2 = 36(1 - \cos^2\theta)$ or $9x^2 - 12xy\cos\theta + 4y^2 = 36\sin^2\theta$.

Ex.16 Let $\alpha = \sin^{-1}\left(\frac{36}{85}\right)$, $\beta = \cos^{-1}\left(\frac{4}{5}\right)$ and $\gamma = \tan^{-1}\left(\frac{8}{15}\right)$, find $(\alpha + \beta + \gamma)$ and hence prove that

(i) $\sum\cot\alpha = \prod\cot\alpha$, **(ii)** $\sum\tan\alpha\cdot\tan\beta = 1$

Sol. $\alpha + \beta + \gamma = \tan^{-1}\left(\frac{36}{77}\right) + \tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{8}{15}\right)$ ....(1)

now $\tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{8}{15}\right) = \tan^{-1}\left(\frac{(3/4)+(8/15)}{1-(24/60)}\right) = \tan^{-1}\left(\frac{77}{36}\right)$

hence $\alpha + \beta + \gamma = \tan^{-1}\left(\frac{36}{77}\right) + \cot^{-1}\left(\frac{36}{77}\right) = \frac{\pi}{2}$

now $\alpha + \beta = \frac{\pi}{2} - \gamma \quad \cot(\alpha + \beta) = \cot\left(\frac{\pi}{2} - \gamma\right) = \tan\gamma = \frac{1}{\cot\gamma}$

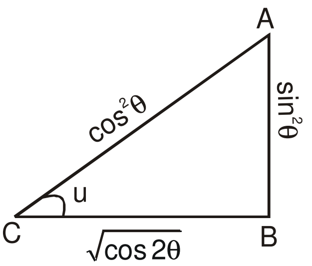
$\frac{\cot\alpha\cot\beta - 1}{\cot\alpha + \cot\beta} = \frac{1}{\cot\gamma} \qquad \therefore \qquad \sum\cot\alpha = \prod\cot\alpha$ hence proved.

|||ly $\tan(\alpha + \beta) = \frac{1}{\tan\gamma}$; $\frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \frac{1}{\tan\gamma}$; $\qquad \therefore \sum\tan\alpha\tan\beta = 1$

Ex.17 If $u = \cos^{-1}\sqrt{\cos 2\theta} - \tan^{-1}\sqrt{\cos 2\theta}$, prove that $\sin u = \tan^2\theta$

Sol. Given, $u = \tan^{-1}\frac{1}{\sqrt{\cos 2\theta}} - \tan^{-1}\sqrt{\cos 2\theta} = \tan^{-1}\left[\frac{\frac{1}{\sqrt{\cos 2\theta}} - \sqrt{\cos 2\theta}}{1 + \frac{1}{\sqrt{\cos 2\theta}}.\sqrt{\cos 2\theta}}\right] = \tan^{-1}\left[\frac{1 - \cos 2\theta}{2\sqrt{\cos 2\theta}}\right]$

$= \tan^{-1}\frac{\sin^2\theta}{\sqrt{\cos 2\theta}} \qquad \therefore \qquad \tan u = \frac{\sin^2\theta}{\sqrt{\cos 2\theta}} = \frac{AB}{BC}$ (say)



Then $AC = \sqrt{\sin^4\theta + \cos 2\theta} = \sqrt{\sin^4\theta + 1 - 2\sin^2\theta} = \cos^2\theta$

$\therefore \sin u = \frac{AB}{AC} = \frac{\sin^2\theta}{\cos^2\theta} = \tan^2\theta$.

Ex.18 Show that $\cos^{-1}\frac{1}{7} + 2\cot^{-1}\frac{1}{3} = \frac{5\pi}{4}$

Sol. $\cot^{-1}\frac{1}{7} + 2\cot^{-1}\frac{1}{3} = \frac{\pi}{2} - \tan^{=1}\frac{1}{7} + 2\left(\frac{\pi}{2} - \tan^{-1}\frac{1}{3}\right)$

$= \frac{3\pi}{2} - \left(\tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{1}{3}\right) = \frac{3\pi}{2} - \left(\tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{2.1/3}{1-(1/3)^2}\right) \quad \left(\because \frac{1}{3} < 1\right)$

$= \frac{3\pi}{2} - \left(\tan^{-1}\frac{1}{7} + \tan^{-1}\frac{3}{4}\right) = \frac{3\pi}{2} - \tan^{-1}\frac{\frac{1}{7} + \frac{3}{4}}{1 - \frac{1}{7}.\frac{3}{4}} \quad \left(\because \frac{1}{7}.\frac{3}{4} < 1\right)$

$= \frac{3\pi}{2} - \tan^{-1}1 = \frac{3\pi}{2} - \frac{\pi}{4} = \frac{5\pi}{4}$

Ex.19 Evaluate **(i)** $\tan\left(\frac{1}{2}\cos^{-1}\frac{2}{3}\right)$ **(ii)** $\tan\left(2\tan^{-1}\frac{1}{5}-\frac{\pi}{4}\right)$

Sol. **(i)** Let $\cos^{-1}\frac{2}{3} = \theta$. Then $\cos\theta = \frac{2}{3}$ and $0 < \theta < \frac{\pi}{2}$

Now $\tan\left(\frac{1}{2}\cos^{-1}\frac{2}{3}\right) = \tan\left(\frac{1}{2}\theta\right) = \frac{1-\cos\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sqrt{1-\cos^2\theta}} = \frac{1-2/3}{\sqrt{1-(2/3)^2}} = \frac{1/3}{\frac{\sqrt{9-4}}{3}} = \frac{1}{\sqrt{5}}$

Alternatively : $\tan^2\frac{\theta}{2} = \frac{1-\cos\theta}{1+\cos\theta} = \frac{1-2/3}{1+2/3} = \frac{1}{5}$ $\quad\therefore \tan\frac{\theta}{2} = \frac{1}{\sqrt{5}}$. $\left[\text{As } 0 < \frac{\theta}{2} < \frac{\pi}{4} < \frac{\pi}{2}\right]$

(ii) Let $\tan^{-1}\frac{1}{5} = \theta$. Then $\tan\theta = \frac{1}{5}$ and $0 < \theta < \frac{\pi}{2}$

Now $\tan\left(2\tan^{-1}\frac{1}{5}-\frac{\pi}{4}\right) = \tan\left(2\theta-\frac{\pi}{4}\right) = \frac{\tan 2\theta - 1}{1+\tan 2\theta} = \frac{\frac{2\tan\theta}{1-\tan^2\theta}-1}{1+\frac{2\tan\theta}{1-\tan^2\theta}}$

$$= \frac{2\tan\theta - 1 + \tan^2\theta}{1-\tan 2\theta + 2\tan\theta} = \frac{\frac{2}{5}-1+\frac{1}{25}}{1-\frac{1}{25}+\frac{2}{5}} = \frac{10-25+1}{25-1+10} = \frac{-14}{34} = \frac{-7}{17}.$$

Ex.20 Evaluate $\cos\left(\frac{1}{2}\arccos\left(-\frac{1}{10}\right)\right)$.

Sol. Let $\alpha = \arccos\left(-\frac{1}{10}\right)$, then $\cos\alpha = -\frac{1}{10}$, $0 < \alpha < \pi$ and $\frac{\alpha}{2}$ lies in the first quadrant.

We get $\cos\left(\frac{1}{2}\arccos\left(-\frac{1}{10}\right)\right) = \sqrt{\frac{1+\cos\alpha}{2}} = \sqrt{\frac{9}{20}} = \frac{3\sqrt{5}}{10}$.

Ex.21 Evaluate $\cos\left(3\arccos\left(-\frac{1}{3}\right)\right)$.

Sol. Let $\alpha = \arccos\left(-\frac{1}{3}\right)$. then $\cos\alpha = -\frac{1}{3}$. we make use of identity to get

$\cos 3a = 4\cos^3 a - 3\cos a = 4\left(-\frac{1}{27}\right) - 3\left(-\frac{1}{3}\right) = 1 - \frac{4}{27} = \frac{23}{27}$

Ex.22 Evaluate $\sin\left(\frac{1}{2}\cot^{-1}\left(\frac{-3}{4}\right)\right)$

Sol. Let $\cot^{-1}\left(\frac{-3}{4}\right) = \theta$. Then $\cot\theta = \frac{-3}{4}$ and $\frac{\pi}{2} < \theta < \pi$.

Thus $\sin\left(\frac{1}{2}\cot^{-1}\left(\frac{-3}{4}\right)\right) = \sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{1-\cos\theta}{2}} = \frac{1}{\sqrt{2}}\sqrt{1-\frac{\cos\theta}{\sin\theta\,\mathrm{cosec}\,\theta}} = \frac{1}{\sqrt{2}}\sqrt{1-\frac{\cot\theta}{\mathrm{cosec}\,\theta}}$

$$= \frac{1}{\sqrt{2}}\sqrt{1-\frac{\cot\theta}{\sqrt{1+\cot^2\theta}}} \qquad [\therefore \text{ for } \pi/2 < \theta < \pi,\ \mathrm{cosec}\,\theta > 0]$$

$$= \frac{1}{\sqrt{2}}\sqrt{1-\frac{-3/4}{\sqrt{1+9/16}}} = \frac{1}{\sqrt{2}}\sqrt{1+\frac{3}{5}} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}.$$

Ex.23 Prove that, $\cos^{-1}\left[1-\frac{1-\cos x}{12\cos x+13}\right] = \pi - 2\cot^{-1}\left(\frac{1}{5}\tan\frac{x}{2}\right)$.

Sol. L. H. S. $= \cos^{-1}\left[\frac{13\cos x+12}{12\cos x+13}\right] = \cos^{-1}\left[\frac{12(1+\cos x)+\cos x}{12(1+\cos x)+1}\right]$

$$= \cos^{-1}\left[\frac{24\cos^2\frac{x}{2}+\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{24\cos^2\frac{x}{2}+\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}\right] = \cos^{-1}\left[\frac{25\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{25\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}\right]$$

$$= \cos^{-1}\left[\frac{1-\left(\frac{1}{5}\tan\frac{x}{2}\right)^2}{1+\left(\frac{1}{5}\tan\frac{x}{2}\right)^2}\right] = 2\tan^{-1}\left(\frac{1}{5}\tan\frac{x}{2}\right) = 2\left[\frac{\pi}{2}-\cot^{-1}\left(\frac{1}{5}\tan x\right)\right] = \pi - 2\cot^{-1}\left(\frac{1}{5}\tan x\right)$$

Ex.24 Prove that $\tan\left(\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\frac{a}{b}\right) + \tan\left(\frac{\pi}{4}-\frac{1}{2}\cos^{-1}\frac{a}{b}\right) = \frac{2b}{a}$.

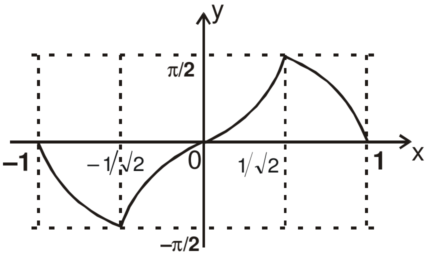
Sol. Let $\cos^{-1}\frac{a}{b} = \theta \qquad \therefore \cos\theta = \frac{a}{b}$

$$\text{L.H.S.} = \tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right) + \tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right) = \frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}} + \frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}$$

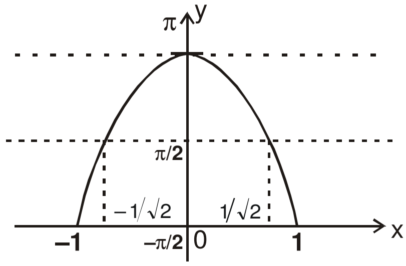
$$= \frac{\left(1+\tan\frac{\theta}{2}\right)^2+\left(1-\tan\frac{\theta}{2}\right)^2}{1-\tan^2\frac{\theta}{2}} = \frac{2\left(1+\tan^2\frac{\theta}{2}\right)}{1-\tan^2\frac{\theta}{2}} = \frac{2}{\cos\theta} = \frac{2b}{a} = \text{R.H.S.}$$

E. SIMPLIFIED INVERSE TRIGONOMETRIC FUNCTIONS

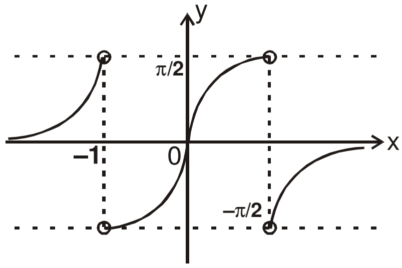
(i) $\sin^{-1} 2x\sqrt{1-x^2} = \begin{cases} -\pi - 2\sin^{-1}x & ; -1 \le x \le \dfrac{-1}{\sqrt{2}} \\ 2\sin^{-1}x & ; \dfrac{-1}{\sqrt{2}} \le x \le \dfrac{1}{\sqrt{2}} \\ \pi - 2\sin^{-1}x & ; \dfrac{1}{\sqrt{2}} \le x \le 1 \end{cases}$



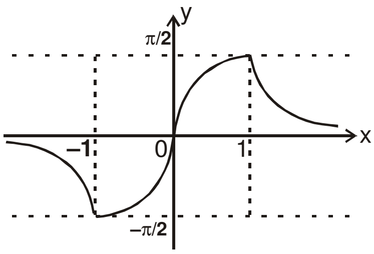
(ii) $\cos^{-1}(2x^2 - 1) = \begin{cases} 2\cos^{-1}x & ; 0 \le x \le 1 \\ 2\pi - 2\cos^{-1}x & ; -1 \le x \le 0 \end{cases}$



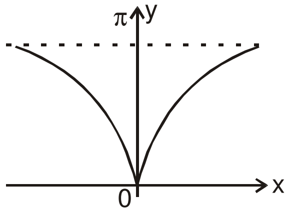
(iii) $\tan^{-1}\dfrac{2x}{1-x^2} = \begin{cases} \pi + 2\tan^{-1}x & ; x < -1 \\ 2\tan^{-1}x & ; -1 < x < 1 \\ -\pi + 2\tan^{-1}x & ; x > 1 \end{cases}$



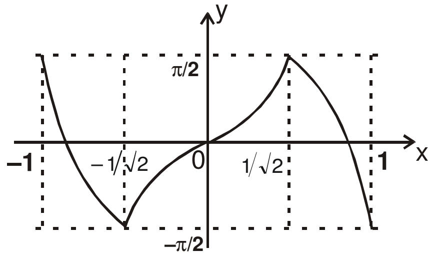
(iv) $\sin^{-1}\dfrac{2x}{1+x^2} = \begin{cases} -\pi - 2\tan^{-1}x & ; x < -1 \\ 2\tan^{-1}x & ; -1 \le x \le 1 \\ \pi - 2\tan^{-1}x & ; x > 1 \end{cases}$



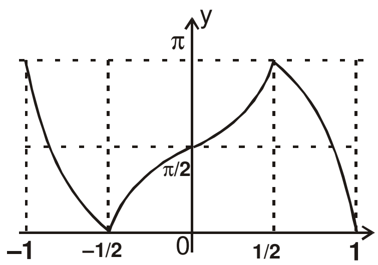
(v) $\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) = \begin{cases} 2\tan^{-1}x & ; x \ge 0 \\ -2\tan^{-1}x & ; x \le 0 \end{cases}$



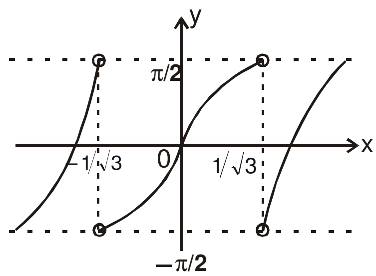
(vi) $\sin^{-1}(3x - 4x^3) = \begin{cases} -\pi - 3\sin^{-1}x & ; -1 \le x \le \dfrac{-1}{2} \\ 3\sin^{-1}x & ; -\dfrac{1}{2} \le x \le \dfrac{1}{2} \\ \pi - 3\sin^{-1}x & ; \dfrac{1}{2} \le x \le 1 \end{cases}$



(vii) $\cos^{-1}(4x^3 - 3x) = \begin{cases} -2\pi + 3\cos^{-1}x & ; \ -1 \le x \le \frac{-1}{2} \\ 2\pi - 3\cos^{-1}x & ; \ -\frac{1}{2} \le x \le \frac{1}{2} \\ 3\cos^{-1}x & ; \ \frac{1}{2} \le x \le 1 \end{cases}$



(viii) $\tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right) = \begin{cases} \pi + 3\tan^{-1}x & ; \ x < \frac{-1}{\sqrt{3}} \\ 3\tan^{-1}x & ; \ \frac{-1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}} \\ -\pi + 3\tan^{-1}x & ; \ x > \frac{1}{\sqrt{3}} \end{cases}$



F. EQUATIONS INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

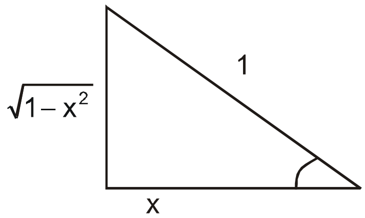
Ex.25 Solve $\cos^{-1} x\sqrt{3} + \cos^{-1}x = \pi/2$.

Sol. Given, $\cos^{-1} x\sqrt{3} + \cos^{-1} x = \pi/2$ ...(1)

or $\cos^{-1} x\sqrt{3} = \pi/2 - \cos^{-1} x$

or $\cos\cos^{-1} x\sqrt{3} = \cos(\pi/2 - \cos^{-1}x)$

or $x\sqrt{3} = \sin\cos^{-1}x$ or $x\sqrt{3} = \sin\sin^{-1}\sqrt{1-x^2}$



or $x\sqrt{3} = \sqrt{1-x^2}$ Squaring we get $3x^2 = 1 - x^2$ or $4x^2 = 1 \Rightarrow x = \pm 1/2$

Verification : When x = 1/2

L.H.S. of equation = $\cos^{1}(\sqrt{3}/2) + \cos^{-1}(1/2) = \pi/6 + \pi/3 + \pi/2$ = R.H.S. of equation

When x = –1/2.

L.H.S. of equation = $\cos^{-1}(-\sqrt{3}/2) + \cos^{-1}(-1/2) = \pi - \cos^{-1}(\sqrt{3}/2) + \pi - \cos^{-1}(1/2)$

$= \pi - \pi/6 + \pi - \pi/3 = 3\pi/2 \neq$ R.H.S. of equation $\quad \therefore \quad$ x = 1/2 is the only solution

Ex.26 Solve for x : $(\tan^{-1} x)^2 + (\cot^{-1} x)^2 = \frac{5\pi^2}{8}$.

Sol. We have $(\tan^{-1} x)^2 + (\cot^{-1} x)^2 = \frac{5\pi^2}{8}$

$\Rightarrow \quad (\tan^{-1} x + \cot^{-1} x)^2 - 2\tan^{-1} x.\cot^{-1} x = \frac{5\pi^2}{8} \quad \Rightarrow \left(\frac{\pi}{2}\right)^2 - 2\tan^{-1} x.(\pi/2 - \tan^{-1} x) = \frac{5\pi^2}{8}$

$\Rightarrow \frac{\pi^2}{4} - 2.\frac{\pi}{2}.\tan^{-1}x + 2(\tan^{-1}x)^2 = \frac{5\pi^2}{8}$ $\Rightarrow 2(\tan^{-1}x)^2 - \pi\tan^{-1}x - \frac{3\pi^2}{8} = 0$

$\Rightarrow \tan^{-1}x = -\pi/4, 3\pi/4 \Rightarrow \tan^{-1}x = -\pi/4;\ x = -1$ {neglecting $\tan^{-1}x = 3\pi/4$ as $\tan^{-1}x \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ }

Ex.27 Determine the integral values of 'k' for which the system, $(\text{arc}\tan x)^2 + (\text{arc}\cos y)^2 = \pi^2 k$ and $\tan^{-1}x + \cos^{-1}y = \frac{\pi}{2}$ posses solution and find all the solutions.

Sol. $\left.\begin{array}{l} 0 \le (\tan^{-1}x)^2 \le \frac{\pi^2}{4} \\ 0 \le (\cos^{-1}y)^2 \le \pi^2 \end{array}\right] \Rightarrow (\tan^{-1}x)^2 + (\cos^{-1}x)^2 \le \frac{5\pi^2}{4}$

But $(\tan^{-1}x)^2 + (\cos^{-1}x)^2 = \pi^2 k$ hence $k\pi^2 \le \frac{5\pi^2}{4} \Rightarrow k \le \frac{5}{4}$ ...(1)

Now put $\tan^{-1}x = \frac{\pi}{2} - \cos^{-1}y \Rightarrow \left(\frac{\pi}{2} - \cos^{-1}y\right)^2 + (\cos^{-1}y)^2 = \pi^2 k$ (where $\cos^{-1}y = t$)

$\Rightarrow 2t^2 - \pi t + \left(\frac{\pi^2}{4} - k\pi^2\right) = 0$ For real roots $D \ge 0$

$\Rightarrow \pi^2 - 8\left(\frac{\pi^2}{4} - k\pi^2\right) \ge 0 \Rightarrow 1 - 2 + 8k \ge 0 \Rightarrow k \ge \frac{1}{8}$ ...(2)

From (1) and (2) $k = 1$

$\therefore t = \frac{\pi \pm \sqrt{8\pi^2 - \pi^2}}{4} = \frac{\pi \pm \sqrt{7}\pi}{4} = \left(1 \pm \sqrt{7}\right)\frac{\pi}{4}$ or $\cos^{-1}y = \left(\sqrt{7}+1\right)\frac{\pi}{4}$ (as $0 \le \cos^{-1}y \le \pi$)

$\therefore \tan^{-1}x = \frac{\pi}{2} - \left(\sqrt{7}+1\right)\frac{\pi}{4} = \frac{\pi}{4}\left[\left(1-\sqrt{7}\right)\right] \Rightarrow x = \tan\left(1-\sqrt{7}\right)\frac{\pi}{4}$

Ex.28 Solve $\tan^{-1}2x + \tan^{-1}3x = 3\pi/4$.

Sol. **(i)** For $x \le 0$, $\tan^{-1}3x \le 0$, therefore $x \le 0$ gives no solution.

(ii) For $x > 0$ if $2x . 3x = 6x^2 <$ i.e. $x < \frac{1}{\sqrt{6}}$ then $\frac{3\pi}{4} = \tan^{-1}2x + \tan^{-1}3x < \frac{\pi}{2}$ which is possible

(iii) For $x > 0$ if $2x . 3x > 1$ i.e. $x > \frac{1}{\sqrt{6}}$ then $\frac{3\pi}{4} = \tan^{-1}x + \tan^{-1}3x = \pi + \tan^{-1}\frac{5x}{1-6x^2}$

i.e. $\tan^{-1}\dfrac{5x}{1-6x^2} = \dfrac{-\pi}{4} \Rightarrow \dfrac{5x}{1-6x^2} = \tan\left(\dfrac{-\pi}{4}\right) = -1 \Rightarrow 5x = -1 + 6x^2$ or $6x^2 - 5x - 1 = 0$ or $x = 1$

As $1 > \dfrac{1}{\sqrt{6}}$ $\therefore$ $x = 1$ is a solution. $x = -\dfrac{1}{6}$ (being rejected)

Ex.29 Solve $\sin^{-1}\dfrac{5x}{13} + \sin^{-1}\dfrac{12x}{13} = \sin^{-1} x$ for x.

Sol. $\sin^{-1}\dfrac{5x}{13} + \sin^{-1}\dfrac{12x}{13} = \sin^{-1} x \Rightarrow \sin\left(\sin^{-1}\dfrac{5x}{13} + \sin^{-1}\dfrac{12x}{13}\right) = \sin(\sin^{-1} x) = x$

$\Rightarrow \dfrac{5x}{13} \cdot \sqrt{1-\left(\dfrac{12x}{13}\right)^2} + \dfrac{12x}{13}\sqrt{1-\left(\dfrac{5x}{13}\right)^2} = x \Rightarrow 5x\sqrt{169-144x^2} + 12x\sqrt{169-25x^2} = 169x$

the above equation holds if either $x = 0$ or $5\sqrt{169-144x^2} + 12x\sqrt{169-25x^2} = 169x$

i.e. $5\sqrt{169-144x^2} = 169 - 12\sqrt{169-25x^2} = 169$

i.e. $25(169 - 144x^2) = 169^2 - 169 \times 24\sqrt{169-25x^2} + 144(169 - 25x^2)$

i.e. $25 = 169 + 144 - 24\sqrt{169-25x^2}$ i.e. $24\sqrt{169-25x^2} = 288$

i.e. $169 - 25x^2 = 144$ i.e. $x^2 = 1$ i.e. $x = \pm 1$

$x = 0, \pm 1$ all are solutions of the equation. Check yourself.

G. INEQUATIONS INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

Ex.30 Find the interval in which $\cos^{-1} x > \sin^{-1} x$.

Sol. We have, $\cos^{-1} x > \sin^{-1}$ {for $\cos^{-1} x$ to be real; $x \in [-1, 1]$}

$\Rightarrow 2\cos^{-1} x > \pi/2 \Rightarrow \cos^{-1} x > \pi/4$ or $\cos(\cos^{-1} x) < \cos \pi/4$

$\Rightarrow x < \dfrac{1}{\sqrt{2}}$ $\therefore$ $x \in \left(-1, \dfrac{1}{\sqrt{2}}\right)$

Ex.31 Find the solution set of the inequation $\sin^{-1}(\sin 5) > x^2 - 4x$

Sol. $\sin^{-1}(\sin 5) > x^2 - 4x \Rightarrow \sin^{-1}[\sin(5 - 2\pi)] > x^2 - 4x$

$\Rightarrow x^2 - 4x < 5 - 2\pi \Rightarrow x^2 - 4x + (2\pi - 5) < 0$

$\Rightarrow 2 - \sqrt{9-2\pi} < x < 2 + \sqrt{9-2\pi} \Rightarrow x \in \left(2-\sqrt{9-2\pi}, 2+\sqrt{9-2\pi}\right)$

H. SUMMATION OF SERIES

Ex.32 Sum the series, $\tan^{-1}\dfrac{4}{1+3\cdot4} + \tan^{-1}\dfrac{6}{1+8\cdot9} + \tan^{-1}\dfrac{8}{1+15\cdot16} + \ldots\ldots$ to 'n' terms.

Sol. $T_n = \tan^{-1}\dfrac{2(n+1)}{1+\{(n+1)^2-1\}\{(n+1)^2\}} = \tan^{-1}\dfrac{2n+2}{1+(n^2+2n)(n+1)^2}$

$= \tan^{-1}\left[\dfrac{2n+2}{1+n(n+2)(n+1)(n+1)}\right] = \tan^{-1}\left[\dfrac{(n+1)(n+2)-n(n+1)}{1+\{n(n+1)\}\{(n+1)(n+2)\}}\right]$

$= \tan^{-1}(n+1)(n+2) - \tan^{-1} n(n+1)$

Put n = 1, 2, 3, ..., n and add, we get $S_n = \tan^{-1}(n+1)(n+2) - \tan^{-1} 2$

Ex.33 Sum the series to 'n' terms, $\tan^{-1}\dfrac{2}{4} + \tan^{-1}\dfrac{2}{9} + \tan^{-1}\dfrac{2}{16} + \tan^{-1}\dfrac{2}{25} + \ldots\ldots$ to 'n' terms.

Also show that, $S_\infty = \tan^{-1} 3$.

Sol. $T_n = \tan^{-1}\dfrac{2}{n^2+2n+1} = \tan^{-1}\dfrac{(n+2)-n}{1+n(n+2)} = \tan^{-1}(n+2) - \tan^{-1}(n)$

Hence, $S_n = \tan^{-1}(n+2) + \tan^{-1}(n+1) - (\tan^{-1}1 + \tan^{-1}2)$

$$S_\infty = \lim_{n\to\infty} S_n = \frac{\pi}{2}+\frac{\pi}{2}-\left(\pi+\tan^{-1}\left(\frac{1+2}{1-2}\right)\right) = \tan^{-1} 3$$

Ex.34 If the sum $\displaystyle\sum_{n=1}^{10}\sum_{m=1}^{10}\tan^{-1}\left(\frac{m}{n}\right) = k\pi$, find the value of k.

Sol. $S = \displaystyle\sum_{n=1}^{10}\left(\tan^{-1}\frac{1}{n}+\tan^{-1}\frac{2}{n}+\tan^{-1}\frac{3}{n}+\ldots\ldots+\tan^{-1}\frac{10}{n}\right)$ now consider

$$\sum_{n=1}^{10}\tan^{-1}\frac{1}{n} = \underbrace{\tan^{-1}1}+\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}+\ldots\ldots+\tan^{-1}\frac{1}{9}+\tan^{-1}\frac{1}{10}$$

$$\sum_{n=1}^{10}\tan^{-1}\frac{2}{n} = \tan^{-1}\frac{2}{1}+\underbrace{\tan^{-1}\frac{2}{2}}+\tan^{-1}\frac{2}{3}+\tan^{-1}\frac{2}{4}+\tan^{-1}\frac{2}{5}+\ldots\ldots+\tan^{-1}\frac{2}{10} \quad \ldots\ldots\ldots$$

$$\sum_{n=1}^{10}\tan^{-1}\frac{10}{n} = \tan^{-1}\frac{10}{1}+\tan^{-1}\frac{10}{2}+\tan^{-1}\frac{10}{3}+\tan^{-1}\frac{10}{4}+\ldots\ldots+\underbrace{\tan^{-1}\frac{10}{10}}$$

$$S = \underbrace{\left(10\cdot\frac{\pi}{4}\right)} + \underbrace{\left(\tan^{-1}\frac{1}{2}+\tan^{-1}2\right)} + \left(\tan^{-1}\frac{1}{3}+\tan^{-1}3\right) + \left(\tan^{-1}\frac{1}{4}+\tan^{-1}4\right) + \ldots\ldots$$

[45 such pair each pair have value equal to $\pi/2$]

$S = \dfrac{5\pi}{2} + \dfrac{45\pi}{2} = \dfrac{50\pi}{2} = 25\pi \quad \Rightarrow \quad k = 25$

Ex.35 Find the value of 'x' satisfying the equation,

$$2\tan^{-1}[\operatorname{cosec}\tan^{-1}x - \cot\tan^{-1}x] = 2\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{1}{8} \quad (\text{given } x \neq 0).$$

Sol. $\text{LHS} = 2\tan^{-1}\left[\operatorname{cosec}\operatorname{cosec}^{-1}\frac{\sqrt{x^2+1}}{x} - \cot\cot^{-1}\frac{1}{x}\right]$

Case-I : If $x > 0$ then $\text{LHS} = 2\tan^{-1}\left[\frac{\sqrt{x^2+1}}{x} - \frac{1}{x}\right] = 2\tan^{-1}\left[\frac{\sqrt{x^2+1}-1}{x}\right] = 2\tan^{-1}\frac{\sec\theta - 1}{\tan\theta}$

$= 2\tan^{-1}\tan\frac{\theta}{2} = \theta = \tan^{-1}x\,[\because x = \tan\theta \Rightarrow \theta = \tan^{-1}x\,;\ \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)]$

Case-II : If $x < 0$ then LHS $= \cot\tan^{-1}x = \cot\left(\cot^{-1}\frac{1}{x} - \pi\right) = -\cot\left(\pi - \cot^{-1}\frac{1}{x}\right) = \cot\cot^{-1}\left(\frac{1}{x}\right) = \frac{1}{x}$

which gives the same result. Hence LHS $= \tan^{-1}x$

$\text{RHS} = 2\left(\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8}\right) = 2\left(\tan^{-1}\frac{\frac{1}{5}+\frac{1}{8}}{1-\frac{1}{40}}\right) = 2\tan^{-1}\frac{1}{3} = \tan^{-1}\frac{2/3}{1-\frac{1}{9}} = \tan^{-1}\frac{3}{4}$

Now $\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{28}} = \tan^{-1}1$. Hence $\tan^{-1}x = \tan^{-1}1 \Rightarrow x = 1$

Ex.36 Solve for x : If $[\sin^{-1}\cos^{-1}\sin^{-1}\tan^{-1}x] = 1$ (where $[\ast]$ denotes the greatest integer function)

Sol. We have; $[\sin^{-1}\cos^{-1}\sin^{-1}\tan^{-1}x] = 1 \Rightarrow 1 \le \sin^{-1}.\cos^{-1}.\sin^{-1}.\tan^{-1}x \le \frac{\pi}{2}$

$\Rightarrow \sin 1 \le \cos^{-1}.\sin^{-1}.\tan^{-1}x \le 1 \Rightarrow \cos\sin 1 \ge \sin^{-1}.\tan^{-1}x \ge \cos 1$

$\Rightarrow \sin\cos\sin 1 \ge \tan^{-1}x \ge \sin\cos 1 \Rightarrow \tan\sin\cos\sin 1 \ge x \ge \tan\sin\cos 1$

Hence, $x \in [\tan\sin\cos 1, \tan\sin\cos\sin 1]$

EXERCISE – I SINGLE CORRECT (OBJECTIVE QUESTIONS)

1. Domain of $f(x) = \cos^{-1} x + \cot^{-1} x + \operatorname{cosec}^{-1} x$ is
(A) $[-1, 1]$ (B) R (C) $(-\infty, -1] \cup [1, \infty)$ (D) $\{-1, 1\}$
Sol.

2. $\operatorname{cosec}^{-1} (\cos x)$ is real if
(A) $x \in [-1, 1]$ (B) $x \in R$
(C) x is an odd multiple of $\pi/2$ (D) x is a multiple of π
Sol.

3. Range of $f(x) = \sin^{-1} x + \tan^{-1} x + \sec^{-1} x$ is
(A) $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$ (B) $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$
(C) $\left\{\frac{\pi}{4}, \frac{3\pi}{4}\right\}$ (D) None of these
Sol.

4. If $x \geq 0$ and $\theta = \sin^{-1} x + \cos^{-1} x - \tan^{-1}x$, then
(A) $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{4}$ (B) $0 < \theta \leq \frac{\pi}{4}$
(C) $0 < \theta \leq \frac{\pi}{2}$ (D) $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$
Sol.

5. If $\cos [\tan^{-1} \{\sin (\cot^{-1} \sqrt{3})\}] = y$, then :
(A) $y = \frac{4}{5}$ (B) $y = \frac{2}{\sqrt{5}}$ (C) $y = -\frac{2}{\sqrt{5}}$ (D) $y^2 = \frac{10}{11}$
Sol.

6. The value of $\tan\left[\sin^{-1}\left(\frac{3}{5}\right)+\tan^{-1}\left(\frac{2}{3}\right)\right]$ is

(A) $\frac{6}{17}$ (B) $\frac{7}{16}$ (C) $\frac{5}{7}$ (D) $\frac{17}{6}$

Sol.

7. If $\sum_{i=1}^{n}\cos^{-1}\alpha_i = 0$, then $\sum_{i=1}^{n}\alpha_i$

(A) n (B) −n (C) 0 (D) None of these

Sol.

8. $\tan\left(\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x\right) + \tan\left(\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x\right)$, $x \neq 0$ is equal to

(A) x (B) 2x (C) $\frac{2}{x}$ (D) $\frac{x}{2}$

Sol.

9. If $\cot^{-1}\frac{n}{\pi} > \frac{\pi}{6}$, $n \in N$, then the maximum value of 'n' is

(A) 1 (B) 5 (C) 9 (D) None of these

Sol.

10. The set of values of 'a' for which
$x^2 + ax + \sin^{-1}(x^2 - 4x + 5) + \cos^{-1}(x^2 - 4x + 5) = 0$
has at least one solution is

(A) $(-\infty, -\sqrt{2\pi}] \cup [\sqrt{2\pi}, \infty)$ (B) $(-\infty, -\sqrt{2\pi}) \cup (\sqrt{2\pi}, \infty)$

(C) R (D) None of these

Sol.

11. If $[\cot^{-1} x] + [\cos^{-1} x] = 0$ then complete set of values of 'x' is
(where [*] denotes the greatest integer function)

(A) (cos 1, 1] (B) (cot 1, cos 1)

(C) (cot 1, 1] (D) None of these

Sol.

12. The value of $\cos\left(\frac{1}{2}\cos^{-1}\frac{1}{8}\right)$ is equal to

(A) 3/4 (B) –3/4 (C) 1/16 (D) 1/4

Sol.

13. The value of
$\sin^{-1}[\cos\{\cos^{-1}(\cos x) + \sin^{-1}(\sin x)\}]$, where $x \in \left(\frac{\pi}{2}, \pi\right)$ is

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{4}$ (C) $-\frac{\pi}{4}$ (D) $-\frac{\pi}{2}$

Sol.

14. If $\sin^{-1} x + \sin^{-1} y = \frac{2\pi}{3}$, then $\cos^{-1} x + \cos^{-1} y$ is equal to

(A) $\frac{2\pi}{3}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{6}$ (D) π

Sol.

15. If x < 0 then value of $\tan^{-1}(x) + \tan^{-1}\left(\frac{1}{x}\right)$ is equal to

(A) $\frac{\pi}{2}$ (B) $-\frac{\pi}{2}$ (C) 0 (D) None of these

Sol.

16. $\tan^{-1} a + \tan^{-1} b$, where a > 0, b > 0, ab > 1, is equal to

(A) $\tan^{-1}\left(\frac{a+b}{1-ab}\right)$ (B) $\tan^{-1}\left(\frac{a+b}{1-ab}\right) - \pi$

(C) $\pi + \tan^{-1}\left(\frac{a+b}{1-ab}\right)$ (D) $\pi - \tan^{-1}\left(\frac{a+b}{1-ab}\right)$

Sol.

17. The value of $\tan^{-1}\left(\frac{a}{b+c}\right) + \tan^{-1}\left(\frac{b}{c+a}\right)$, if $\angle C = 90^\circ$ in triangle ABC, is

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{2}$ (D) π

Sol.

18. If $\tan^{-1} \frac{\sqrt{1+x^2}-1}{x} = \frac{\pi}{45}$, then

(A) x = tan 2º (B) x = tan 4º

(C) x = tan (1/4)º (D) x = tan 8º

Sol.

19. The value of

$\cot^{-1}\left\{\dfrac{\sqrt{1-\sin x}+\sqrt{1+\sin x}}{\sqrt{1-\sin x}-\sqrt{1+\sin x}}\right\}$, $\dfrac{\pi}{2} < x < \pi$, is

(A) $\pi - \dfrac{x}{2}$ (B) $\dfrac{\pi}{2}+\dfrac{x}{2}$ (C) $\dfrac{x}{2}$ (D) $2\pi - \dfrac{x}{2}$

Sol.

20. The smallest and the largest values of

$\tan^{-1}\left(\dfrac{1-x}{1+x}\right)$, $0 \le x \le 1$ are

(A) $0, \pi$ (B) $0, \dfrac{\pi}{4}$ (C) $-\dfrac{\pi}{4}, \dfrac{\pi}{4}$ (D) $\dfrac{\pi}{4}, \dfrac{\pi}{2}$

Sol.

21. The complete solution set of the inequality $[\cot^{-1} x]^2 - 6[\cot^{-1} x] + 9 \le 0$ is
(where [$*$] denotes the greatest integer function)

(A) $(-\infty, \cot 3]$ (B) $[\cot 3, \cot 2]$
(C) $[\cot 3, \infty)$ (D) None of these

Sol.

22. If $\dfrac{1}{2}\sin^{-1}\left(\dfrac{3\sin 2\theta}{5+4\cos 2\theta}\right)=\dfrac{\pi}{4}$, then $\tan\theta$ is equal to

(A) 1/3 (B) 3 (C) 1 (D) −1

Sol.

23. $\cos^{-1}\left\{\frac{1}{2}x^2 + \sqrt{1-x^2}\cdot\sqrt{1-\frac{x^2}{4}}\right\} = \cos^{-1}\frac{x}{2} - \cos^{-1}x$ holds for

(A) $|x| \le 1$ (B) $x \in R$

(C) $0 \le x \le 1$ (D) $-1 \le x \le 0$

Sol.

24. All possible values of p and q for which

$\cos^{-1}\sqrt{p} + \cos^{-1}\sqrt{1-p} + \cos^{-1}\sqrt{1-q} = \frac{3\pi}{4}$ holds, is

(A) $p = 1,\ q = \frac{1}{2}$ (B) $q > 1,\ p = \frac{1}{2}$

(C) $0 \le p \le 1,\ q = \frac{1}{2}$ (D) None of these

Sol.

25. Which one of the following correct ?

(A) $\tan 1 > \tan^{-1} 1$ (B) $\tan 1 < \tan^{-1} 1$

(C) $\tan 1 = \tan^{-1} 1$ (D) None of these

Sol.

26. If $u = \cot^{-1}\sqrt{\tan\alpha} - \tan^{-1}\sqrt{\tan\alpha}$, then

$\tan\left(\frac{\pi}{4} - \frac{u}{2}\right)$ is equal to

(A) $\sqrt{\tan\alpha}$ (B) $\sqrt{\cot\alpha}$ (C) $\tan\alpha$ (D) $\cot\alpha$

Sol.

27. The equation $\sin^{-1} x - \cos^{-1} x = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ has

(A) no solution (B) unique solution

(C) infinite number of solutions (D) None of these

Sol.

28. The solution of the equation

$\sin^{-1}\left(\tan\frac{\pi}{4}\right) - \sin^{-1}\left(\sqrt{\frac{3}{x}}\right) - \frac{\pi}{6} = 0$ is

(A) x = 2 (B) x = –4 (C) x = 4 (D) None of these

Sol.

29. The set of values of 'x' for which the formula $2\sin^{-1} x = \sin^{-1}(2x\sqrt{1-x^2})$ is true is

(A) (–1, 0) (B) [0, 1]

(C) $\left[-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right]$ (D) $\left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$

Sol.

30. The number of solution(s) of the equation,
$\sin^{-1}x + \cos^{-1}(1 - x) = \sin^{-1}(-x)$, is/are

(A) 0 (B) 1 (C) 2 (D) more than 2

Sol.

31. If $\sin^{-1}\left(\frac{x}{5}\right) + \text{cosec}^{-1}\left(\frac{5}{4}\right) = \frac{\pi}{2}$, then a value of x is

(A) 1 (B) 3 (C) 4 (D) 5

Sol.

32. The value of x satisfying
$\sin^{-1} x + \sin^{-1}(1 - x) = \cos^{-1} x$ are

(A) $0, \frac{1}{2}$ (B) 0 (C) 1, −1 (D) None of these

Sol.

33. The number of solutions of the equation

$\tan^{-1}(1 + x) + \tan^{-1}(1 - x) = \frac{\pi}{2}$ is

(A) 3 (B) 2 (C) 1 (D) 4

Sol.

34. The number of solution(s) of the equation

$\sin^{-1}(1 - x) - 2\sin^{-1} x = \frac{\pi}{2}$, is / are

(A) 0 (B) 1 (C) 2 (D) more than two

Sol.

35. The number of solutions of the equation

$\tan^{-1}\left(\frac{1}{2x+1}\right) + \tan^{-1}\left(\frac{1}{4x+1}\right) = \tan^{-1}\left(\frac{2}{x^2}\right)$ is

(A) 0 (B) 1 (C) 2 (D) 3

Sol.

EXERCISE – II MULTIPLE CORRECT (OBJECTIVE QUESTIONS)

1. If $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \frac{3\pi}{2}$, then

(A) $x^{100} + y^{100} + z^{100} - \frac{9}{x^{101}+y^{101}+z^{101}} = 0$

(B) $x^{22} + y^{42} + z^{62} - x^{220} - y^{420} - z^{620} = 0$

(C) $x^{50} + y^{25} + z^{5} = 0$ (D) $\frac{x^{2008}+y^{2008}+z^{2008}}{(xyz)^{2009}} = 0$

Sol.

2. If α satisfies the inequation $x^2 - x - 2 > 0$, then which of the following exists

(A) $\sin^{-1}\alpha$ (B) $\cos^{-1}\alpha$ (C) $\sec^{-1}\alpha$ (D) $\mathrm{cosec}^{-1}\alpha$

Sol.

3. $6\sin^{-1}\left(x^2 - 6x + \frac{17}{2}\right) = \pi$, if

(A) x = 1 (B) x = 2 (C) x = 3 (D) x = 4

Sol.

4. If the numerical value of $\tan(\cos^{-1}(4/5) + \tan^{-1}(2/3))$ is a/b then

(A) a + b = 23 (B) a – b = 11

(C) 3b = a + 1 (D) 2a = 3b

Sol.

5. The value of $\cos\left[\frac{1}{2}\cos^{-1}\left\{\cos\left(-\frac{14\pi}{5}\right)\right\}\right]$ is

(A) $\cos\left(-\frac{7\pi}{2}\right)$ (B) $\sin\left(\frac{\pi}{10}\right)$

(C) $\cos\left(\frac{2\pi}{5}\right)$ (D) $-\cos\left(\frac{3\pi}{5}\right)$

Sol.

6. If a = $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) + \cos^{-1}\left(-\frac{1}{2}\right)$ and

b = $\tan^{-1}(-\sqrt{3}) - \cot^{-1}\left(-\frac{1}{\sqrt{3}}\right)$, then

(A) $a - b = 17\pi/12$ (B) $a + b = 17\pi/12$
(C) $a + b = -7\pi/12$ (D) $a - b = \pi/12$

Sol.

7. Let $f(x) = \sin^{-1} x + \cos^{-1} x$. Then $\frac{\pi}{2}$ is equal to

(A) $f\left(\frac{1}{2}\right)$ (B) $f(k^2 - 2k + 3)$, $k \in R$
(C) $f\left(\frac{1}{1+k^2}\right)$, $k \in R$ (D) $f(-2)$

Sol.

8. If $\text{cosec}^{-1} x = \sin^{-1}\left(\frac{1}{x}\right)$, then x may be

(A) 1 (B) $-\frac{1}{2}$ (C) $\frac{3}{2}$ (D) $-\frac{3}{2}$

Sol.

9. If $0 < x < 1$, then $\tan^{-1}\frac{\sqrt{1-x^2}}{1+x}$ is equal to

(A) $\frac{1}{2}\cos^{-1} x$ (B) $\cos^{-1}\sqrt{\frac{1+x}{2}}$
(C) $\sin^{-1}\sqrt{\frac{1-x}{2}}$ (D) $\frac{1}{2}\tan^{-1}\sqrt{\frac{1+x}{1-x}}$

Sol.

10. α, β and γ are three angles given by

$\alpha = 2\tan^{-1}(\sqrt{2} - 1)$, $\beta = 3\sin^{-1}\frac{1}{\sqrt{2}} + \sin^{-1}\left(-\frac{1}{2}\right)$

and $\gamma = \cos^{-1}\frac{1}{3}$. Then

(A) $\alpha > \beta$ (B) $\beta > \gamma$ (C) $\alpha < \gamma$ (D) $\alpha > \gamma$

Sol.

Sol.

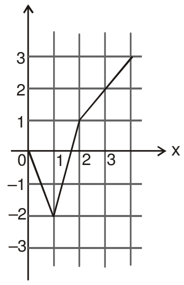
11. For the equation
$2x = \tan(2\tan^{-1} a) + 2\tan(\tan^{-1} a + \tan^{-1} a^3)$,
which of the following is invalid ?
(A) $a^2 x + 2a = x$ (B) $a^2 + 2ax + 1 = 0$
(C) $a \neq 0$ (D) $a \neq -1, 1$
Sol.

12. $\cos^{-1} x = \tan^{-1} x$ then
(A) $x^2 = \left(\frac{\sqrt{5}-1}{2}\right)$ (B) $x^2 = \left(\frac{\sqrt{5}+1}{2}\right)$
(C) $\sin(\cos^{-1} x) = \left(\frac{\sqrt{5}-1}{2}\right)$
(D) $\tan(\cos^{-1} x) = \left(\frac{\sqrt{5}-1}{2}\right)$

13. The sum $\sum_{n=1}^{\infty} \tan^{-1} \frac{4n}{n^4 - 2n^2 + 2}$ is equal to
(A) $\tan^{-1} 2 + \tan^{-1} 3$ (B) $4\tan^{-1} 1$
(C) $\pi/2$ (D) $\sec^{-1}(-\sqrt{2})$
Sol.

EXERCISE – III SUBJECTIVE QUESTIONS

1. Given is partial graph of an even periodic function *f* whose period is 8. If [*] denotes greatest integer function then find the value of the expression.



$$f(-3)+2|f(-1)|+\left[f\left(\frac{7}{8}\right)\right]+f(0)+\text{arc}\cos(f(-2))+f(-7)+f(20)$$

Sol.

2. Solve the following inequalities

(i) $\sin^{-1} x > -1$ **(ii)** $\cos^{-1} x < 2$

(iii) $\cot^{-1} x < -\sqrt{3}$

Sol.

3. Evaluate each of the following

(i) $\sin^{-1}\left(\sin\frac{7\pi}{6}\right)$ **(ii)** $\tan^{-1}\left(\tan\frac{2\pi}{3}\right)$

(iii) $\cos^{-1}\left(\cos\frac{5\pi}{4}\right)$ **(iv)** $\sec^{-1}\left(\sec\frac{7\pi}{4}\right)$

Sol.

4. Find the value of the following

(i) $\sin^{-1}(\sin 5)$ **(ii)** $\cos^{-1}(\cos 10)$

(iii) $\tan^{-1}(\tan(-6))$ **(iv)** $\cot^{-1}(\cot(-10))$

(v) $\cos^{-1}\left(\frac{1}{\sqrt{2}}\left(\cos\frac{9\pi}{10}-\sin\frac{9\pi}{10}\right)\right)$

Sol.

5 . Find $\sin^{-1}(\sin \theta)$, $\cos^{-1}(\cos \theta)$, $\tan^{-1}(\tan \theta)$ and $\cot^{-1}(\cot \theta)$ for $\theta \in \left[\frac{3\pi}{2}, 3\pi\right]$

Sol.

6. Solve the following equations :

(i) $\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\tan^{-1} x$, $(x > 0)$

Sol.

(ii) $3\tan^{-1}\left(\frac{1}{2+\sqrt{3}}\right) - \tan^{-1}\left(\frac{1}{x}\right) = \tan^{-1}\left(\frac{1}{3}\right)$

Sol.

7. Find the value of

$\tan\left\{\frac{1}{2}\sin^{-1}\left(\frac{2x}{1+x^2}\right)+\frac{1}{2}\cos^{-1}\left(\frac{1-y^2}{1+y^2}\right)\right\}$, if x > y > 1.

Sol.

8. Prove each of the following :

(i) $\tan^{-1} x = -\pi + \cot^{-1}\frac{1}{x} = \sin^{-1}\frac{x}{\sqrt{1+x^2}}$

$= -\cos^{-1}\frac{1}{\sqrt{1+x^2}}$ when x < 0.

Sol.

(ii) $\cos^{-1} x = \sec^{-1}\frac{1}{x} = \pi - \sin^{-1}\sqrt{1-x^2}$

$= \pi + \tan^{-1}\frac{\sqrt{1-x^2}}{x} = \cot^{-1}\frac{x}{\sqrt{1-x^2}}$

when $-1 < x < 0$.

Sol.

9. Find the number of values of $x \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ satisfying the equation $\sin^2(2\cos^{-1}(\tan x)) = 1$.

Sol.

10. If $\alpha = 2\tan^{-1}\left(\frac{1+x}{1-x}\right)$ & $\beta = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ for $0 < x < 1$, then prove that $\alpha + \beta = \pi$. What the value of $\alpha + \beta$ will be if $x > 1$?

Sol.

11. If x = sin (2 $\tan^{-1}$ 2) and y = sin $\left(\frac{1}{2}\tan^{-1}\frac{4}{3}\right)$ then find the relation between x and y.

Sol.

12. Prove that

(i) $\cos^{-1}\left(\sqrt{\frac{1}{3}}\right) - \cos^{-1}\left(\sqrt{\frac{1}{6}}\right) + \cos^{-1}\left(\frac{\sqrt{10}-1}{3\sqrt{2}}\right) = \cos^{-1}\left(\frac{2}{3}\right)$

Sol.

(ii) $2\tan^{-1}(\text{cosec } \tan^{-1} x - \tan \cot^{-1} x) = \tan^{-1} x$

Sol.

(iii) $\cos^{-1}\left(\frac{63}{65}\right) + 2\tan^{-1}\left(\frac{1}{5}\right) = \sin^{-1}\left(\frac{3}{5}\right)$

Sol.

13. Find the following

(i) $\tan\left[\cos^{-1}\frac{1}{2}+\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)\right]$

Sol.

(ii) $\sin\left[\frac{\pi}{3}-\sin^{-1}\left(\frac{-1}{2}\right)\right]$

Sol.

(iii) $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$

Sol.

(iv) $\tan^{-1}\left(\tan\frac{2\pi}{3}\right)$

Sol.

(v) $\cos\left(\tan^{-1}\frac{3}{4}\right)$

Sol.

(vi) $\tan\left[\sin^{-1}\frac{3}{5}+\cot^{-1}\frac{3}{2}\right]$

Sol.

14. Find the following

(i) $\sin\left[\frac{\pi}{2}-\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)\right]$

Sol.

(ii) $\cos\left[\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)+\frac{\pi}{6}\right]$

Sol.

(iii) $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$

Sol.

(iv) $\cos^{-1}\left(\cos\frac{4\pi}{3}\right)$

Sol.

(v) $\sin\left[\cos^{-1}\frac{3}{5}\right]$

Sol.

(vi) $\tan^{-1}\left(\frac{3\sin 2\alpha}{5+3\cos 2\alpha}\right) + \tan^{-1}\left(\frac{\tan\alpha}{4}\right)$

where $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$

Sol.

15. Prove that

(i) $2\cos^{-1}\frac{3}{\sqrt{13}} + \cot^{-1}\frac{16}{63} + \frac{1}{2}\cos^{-1}\frac{7}{25} = \pi$

Sol.

(ii) $\cos^{-1}\left(\frac{5}{13}\right) + \cos^{-1}\left(-\frac{7}{25}\right) + \sin^{-1}\frac{36}{325} = \pi$

Sol.

(iii) arc cos $\sqrt{\frac{2}{3}}$ – arc cos $\frac{\sqrt{6}+1}{2\sqrt{3}} = \frac{\pi}{6}$

Sol.

(iv) Solve the inequality :
(arc sec x)^2 – 6(arc sec x) + 8 > 0

Sol.

16. Find the domain of definition the following functions.
(Read the symbols [*] and {*} as greatest integers and fractional part functions respectively.)

(i) f(x) = arc cos $\frac{2x}{1+x}$

Sol.

(ii) $\sqrt{\cos(\sin x)} + \sin^{-1}\frac{1+x^2}{2x}$

Sol.

(iii) f(x) = $\sin^{-1}\left(\frac{x-3}{2}\right) - \log_{10}(4-x)$

Sol.

(iv) $f(x) = \dfrac{\sqrt{1-\sin x}}{\log_5(1-4x^2)} + \cos^{-1}(1 - \{x\})$,

where $\{x\}$ is the fractional part of x.

Sol.

(v) $f(x) = \sqrt{3-x} + \cos^{-1}\left(\dfrac{3-2x}{5}\right) + \log_6(2|x| - 3) + \sin^{-1}(\log_2 x)$

Sol.

(vi) $f(x) = \log_{10}(1 - \log_7(x^2 - 5x + 13)) + \cos^{-1}\left(\dfrac{3}{2+\sin\dfrac{9\pi x}{2}}\right)$

Sol.

(vii) $f(x) = e^{\sin^{-1}\left(\frac{x}{2}\right)} + \tan^{-1}\left[\dfrac{x}{2}-1\right] + ln\left(\sqrt{x-[x]}\right)$

Sol.

(viii) $f(x) = \sqrt{\sin(\cos x)} + ln(-2\cos^2 x + 3\cos x + 1) + e^{\cos^{-1}\left(\frac{2\sin x+1}{2\sqrt{2\sin x}}\right)}$

Sol.

17. Find the solution set of the equation,

$3\cos^{-1} x = \sin^{-1} \left(\sqrt{1-x^2}\ (4x^2 - 1)\right)$.

Sol.

18. Prove that

(i) $\sin^{-1}\cos(\sin^{-1} x) + \cos^{-1}\sin(\cos^{-1} x) = \dfrac{\pi}{2}$, $|x| \le 1$

Sol.

(ii) $2\tan^{-1}(\text{cosec}\tan^{-1}x - \tan\cot^{-1}x) = \tan^{-1}x \ (x \ne 0)$

Sol.

(iii) $\tan^{-1}\left(\dfrac{2mn}{m^2-n^2}\right) + \tan^{-1}\left(\dfrac{2pq}{p^2-q^2}\right) = \tan^{-1}\left(\dfrac{2MN}{M^2-N^2}\right)$

where $M = mp - nq$, $N = np + mq$,

$$\left|\frac{n}{m}\right| < 1; \left|\frac{p}{q}\right| < 1 \text{ and } \left|\frac{N}{M}\right| < 1.$$

Sol.

(iv) $\tan(\tan^{-1}x + \tan^{-1}y + \tan^{-1}z) = \cot(\cot^{-1}x + \cot^{-1}y + \cot^{-1}z)$

Sol.

19. Solve the following equations / system of equations

(i) $\sin^{-1}x + \sin^{-1}2x = \frac{\pi}{3}$

Sol.

(ii) $\tan^{-1}\frac{1}{1+2x} + \tan^{-1}\frac{1}{1+4x} = \tan^{-1}\frac{2}{x^2}$

Sol.

(iii) $\tan^{-1}(x-1) + \tan^{-1}(x) + \tan^{-1}(x+1) = \tan^{-1}(3x)$

Sol.

(iv) $\sin^{-1}\frac{1}{\sqrt{5}} + \cos^{-1}x = \frac{\pi}{4}$

Sol.

(v) $\cos^{-1}\frac{x^2-1}{x^2+1} + \tan^{-1}\frac{2x}{x^2-1} = \frac{2\pi}{3}$

Sol.

(vi) $\sin^{-1}x + \sin^{-1}y = \frac{2\pi}{3}$ & $\cos^{-1}x - \cos^{-1}y = \frac{\pi}{3}$

Sol.

(vii) $2\tan^{-1}x = \cos^{-1}\frac{1-a^2}{1+a^2} - \cos^{-1}\frac{1-b^2}{1+b^2}$ $(a > 0, b > 0)$.

Sol.

20. Let $y = \sin^{-1}(\sin 8) - \tan^{-1}(\tan 10) + \cos^{-1}(\cos 12) - \sec^{-1}(\sec 9) + \cot^{-1}(\cot 6) - \text{cosec}^{-1}(\text{cosec } 7)$. If y simplifies to $a\pi + b$ then find $(a - b)$.

Sol.

21. Show that :

$$\sin^{-1}\left(\sin\frac{33\pi}{7}\right)+\cos^{-1}\left(\cos\frac{46\pi}{7}\right)+\tan^{-1}\left(-\tan\frac{13\pi}{8}\right)+\cot^{-1}\left(\cot\left(-\frac{19\pi}{8}\right)\right)$$

$= \frac{45\pi}{28}$

Sol.

22. If $u = \cot^{-1}\sqrt{\cos 2\theta} - \tan^{-1}\sqrt{\cos 2\theta}$ then prove that $\sin u = \tan^2\theta$.

Sol.

23. If $x \in \left[-1, -\frac{1}{2}\right]$ then express the function $f(x) = \sin^{-1}(3x - 4x^3) + \cos^{-1}(4x^3 - 3x)$ in the form of $a \cos^{-1} x + b\pi$, where a and b are rational numbers.

Sol.

24. Find the sum of the series

(i) $\sin^{-1}\frac{1}{\sqrt{2}} + \sin^{-1}\frac{\sqrt{2}-1}{\sqrt{6}} + \ldots. + \sin^{-1}\frac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n(n+1)}} + \ldots \infty$

Sol.

(ii) $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{2}{9} + \ldots.. + \tan^{-1}\frac{2^{n-1}}{1+2^{2n-1}} + \ldots.. \infty$

Sol.

(iii) $\cot^{-1}7 + \cot^{-1}13 + \cot^{-1}21 + \cot^{-1}31 + \ldots\ldots$ to n terms

Sol.

(iv) $\tan^{-1}\frac{1}{x^2+x+1} + \tan^{-1}\frac{1}{x^2+3x+3} + \tan^{-1}\frac{1}{x^2+5x+7} + \tan^{-1}\frac{1}{x^2+7x+13}$ to n terms.

Sol.

(v) $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{8} + \tan^{-1}\frac{1}{18} + \tan^{-1}\frac{1}{32} + \ldots.. \infty$

Sol.

25. Solve the following

(i) $\cot^{-1}x + \cot^{-1}(n^2 - x + 1) = \cot^{-1}(n - 1)$

Sol.

(ii) $\sec^{-1}\frac{x}{a} - \sec^{-1}\frac{x}{b} = \sec^{-1}b - \sec^{-1}a$ $a \geq 1$; $b \geq 1$, $a \neq b$.

Sol.

(iii) $\tan^{-1}\frac{x-1}{x+1} + \tan^{-1}\frac{2x-1}{2x+1} = \tan^{-1}\frac{23}{36}$

Sol.

26. Solve the following inequalities

(i) $\text{arc}\cot^2 x - 5\,\text{arc}\cot x + 6 > 0$

Sol.

(ii) $\text{arc}\sin x > \text{arc}\cos x$

Sol.

(iii) $\tan^2(\text{arc}\sin x) > 1$

Sol.

EXERCISE – IV ADVANCED SUBJECTIVE QUESTIONS

1. Prove that

(a) $\tan\left[\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\frac{a}{b}\right] + \tan\left[\frac{\pi}{4}-\frac{1}{2}\cos^{-1}\frac{a}{b}\right]=\frac{b}{a}$

Sol.

(b) $\cos^{-1}\frac{\cos x+\cos y}{1+\cos x\cos y}=2\tan^{-1}\left(\tan\frac{x}{2}.\tan\frac{y}{2}\right)$

Sol.

(c) $2\tan^{-1}\left[\sqrt{\frac{a-b}{a+b}}.\tan\frac{x}{2}\right] = \cos^{-1}\left[\frac{b+a\cos x}{a+b\cos x}\right]$

Sol.

2. If $y = \tan^{-1}\left[\dfrac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right]$ prove that

$x^2 = \sin 2y$.

Sol.

3. Find the sum of the series

(a) $\cot^{-1}7 + \cot^{-1}13 + \cot^{-1}21 + \cot^{-1}31 + \ldots\ldots$ to n terms.

Sol.

(b) $\tan^{-1}\dfrac{1}{3} + \tan^{-1}\dfrac{2}{9} + \ldots\ldots + \tan^{-1}\dfrac{2^{n-1}}{1+2^{2n-1}} + \ldots\ldots\ldots\infty$

Sol.

(c) $\tan^{-1}\dfrac{1}{x^2+x+1} + \tan^{-1}\dfrac{1}{x^2+3x+3} + \tan^{-1}\dfrac{1}{x^5+5x+7} + \tan^{-1}\dfrac{1}{x^2+7x+13}$ to n terms.

Sol.

(d) $\sin^{-1}\dfrac{1}{\sqrt{5}} + \sin^{-1}\dfrac{1}{\sqrt{65}} + \sin^{-1}\dfrac{1}{\sqrt{325}} + \ldots\ldots\infty$ terms

Sol.

(e) $\sin^{-1}\frac{1}{\sqrt{2}}+\sin^{-1}\frac{\sqrt{2}-1}{\sqrt{6}}+......+\sin^{-1}\frac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n(n+1)}}+........\infty$

Sol.

4. Express

$\frac{\beta^3}{2}\operatorname{cosec}^2\left[\frac{1}{2}\tan^{-1}\frac{\beta}{\alpha}\right]+\frac{\alpha^3}{2}\sec^2\left[\frac{1}{2}\tan^{-1}\frac{\alpha}{\beta}\right]$

as an integral polynomial in α & β.

Sol.

5. Find the integral value of K for which the system of equations ;

$$\left[\begin{array}{l}\text{arc}\cos x+(\text{arc}\sin y)^2=\frac{K\pi^2}{14}\\(\text{arc}\sin y)^2.(\text{arc}\cos x)=\frac{\pi^4}{16}\end{array}\right.$$

possesses solutions & find those solutions.

Sol.

6. Find all the positive integral solutions of,

$\tan^{-1}x + \cos^{-1}\frac{y}{\sqrt{1+y^2}} = \sin^{-1}\frac{3}{\sqrt{10}}$

Sol.

7. If X = cosec. $\tan^{-1}$. cos. $\cot^{-1}$. sec. $\sin^{-1}a$ & Y = sec $\cot^{-1}$ sin $\tan^{-1}$ cosec $\cos^{-1}$ a ;
where $0 \le a \le 1$. Find the relation between X & Y. Express them in terms of 'a'.

Sol.

8. Column – I **Column – II**

(A) $f(x) = \sin^{-1}\left(\frac{2}{|\sin x - 1| + |\sin x + 1|}\right)$ (P) f(x) is many one

(B) $f(x) = \cos^{-1}(|x-1| - |x-2|)$ (Q) Domain of f(x) is R

(C) $f(x) = \sin^{-1}\left(\frac{\pi}{|\sin^{-1}x - (\pi/2)| + |\sin^{-1}x + (\pi/2)|}\right)$ (R) Range contain only irrational number

(D) $f(x) = \cos(\cos^{-1}|x|)$ (S) f(x) is even.

$+ \sin^{-1}(\sin x) - \operatorname{cosec}^{-1}(\operatorname{cosec} x) + \operatorname{cosec}^{-1}|x|$

Sol.

9. Prove that the equation, $(\sin^{-1}x)^3+(\cos^{-1}x)^3=\alpha\pi^3$ has no roots for $\alpha<\frac{1}{32}$ and $\alpha>\frac{7}{8}$

Sol.

10. Solve the following inequalities

(a) arc $\cot^2 x$ – 5 arc cot x + 6 > 0

Sol.

(b) arc sin x > arc cos x

Sol.

(c) $\tan^2 (\text{arc sin } x) > 1$

Sol.

11. Solve the following system of inequations

$4\text{arc tan}^2x - 8 \text{ arc tan}x + 3 < 0$ &

$4 \text{ arc cot } x - \text{arc cot}^2x - 3 \geq 0$

Sol.

12. If the total area between the curves $f(x) = \cos^{-1}(\sin x)$ ang $g(x) = \sin^{-1}(\cos x)$ on the interval $[-7\pi, 7\pi]$ is A, find the value of 49 A. (Take $\pi = 22/7$)

Sol.

13. If the sum $\sum_{n=1}^{10}\sum_{m=1}^{10}\tan^{-1}\left(\frac{m}{n}\right) = k\pi$, find the value of k.

Sol.

14. Show that the roots r,s and t of the cubic $x(x - 2)(3x - 7) = 2$, are real and positive. Also compute the value of $\tan^{-1}(r) + \tan^{-1}(s) + \tan^{-1}(t)$

Sol.

15. Solve for x : $\sin^{-1}\left(\sin\left(\frac{2x^2+4}{1+x^2}\right)\right) < \pi - 3$

Sol.

16. Find the set of value of 'a' for which the equation $2\cos^{-1}x = a + a^2(\cos^{-1}x)^{-1}$ posses a solution

Sol.

Paragraph question nos. 17 to 19

The domain and range of inverse circular function are defined as follows

	Domain	Range
$\sin^{-1}x$	$[-1, 1]$	$\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$
$\cos^{-1}x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1}x$	R	$\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$
$\cot^{-1}x$	R	$(0, \pi)$
$\text{cosec}^{-1}x$	$(-\infty, -1] \cup [1, \infty)$	$\left[\frac{\pi}{2}, \frac{3\pi}{2}\right] - (\pi)$
$\sec^{-1}x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$

17. $\sin^{-1} x < \frac{3\pi}{4}$ then solution set of x is

(A) $\left(\frac{1}{\sqrt{2}}, 1\right]$ (B) $\left(-\frac{1}{\sqrt{2}}, -1\right]$

(C) $\left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$ (D) None of these

Sol.

18. $\sin^{-1} x + \text{cosec}^{-1} x$ at $x = -1$ is

(A) π (B) 2π (C) 3π (D) $-\pi$

Sol.

19. If $x \in [-1, 1]$, then range of $\tan^{-1}(-x)$ is

(A) $\left[\frac{3\pi}{4}, \frac{7\pi}{4}\right]$ (B) $\left[\frac{3\pi}{4}, \frac{5\pi}{4}\right]$

(C) $[-\pi, 0]$ (D) $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

Sol.

Paragraph question nos. 20 to 22

$$\tan^{-1}(\tan\theta) = \begin{cases} \pi + \theta & , \ -\frac{3\pi}{2} < \theta < -\frac{\pi}{2} \\ \theta & , \ -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ -\pi + \theta & , \ \frac{\pi}{2} < \theta < \frac{3\pi}{2} \end{cases},$$

$$\sin^{-1}(\sin\theta) = \begin{cases} -\pi - \theta & , \ -\frac{3\pi}{2} \le \theta < -\frac{\pi}{2} \\ \theta & , \ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \\ \pi - \theta & , \ \frac{\pi}{2} < \theta \le \frac{3\pi}{2} \end{cases}$$

$$\cos^{-1}(\cos\theta) = \begin{cases} -\theta & , \ -\pi \le \theta < 0 \\ \theta & , \ 0 \le \theta \le \pi \\ 2\pi - \theta & , \ \pi < \theta \le 2\pi \end{cases}$$

Based on the above results, answer each of the following :

20. $\cos^{-1} x$ is equal to

(A) $\sin^{-1} \sqrt{1-x^2}$ if $-1 < x < 1$

(B) $-\sin^{-1} \sqrt{1-x^2}$ if $-1 < x < 0$

(C) $\sin^{-1} \sqrt{1-x^2}$ if $-1 < x < 0$

(D) $\sin^{-1} \sqrt{1-x^2}$ if $0 < x < 1$

Sol.

21. $\sin^{-1} x$ is equal to

(A) $\cos^{-1} \sqrt{1-x^2}$ if $-1 < x < 0$

(B) $\cos^{-1} \sqrt{1-x^2}$ if $-1 < x < 1$

(C) $\cos^{-1} \sqrt{1-x^2}$ if $0 < x < 1$

(D) $-\cos^{-1} \sqrt{1-x^2}$ if $0 < x < 1$

Sol.

22. $\cos^{-1} x$ is equal to

(A) $-\tan^{-1} \dfrac{\sqrt{1-x^2}}{x}$ if $-1 < x < 0$

(B) $-\tan^{-1} \dfrac{\sqrt{1-x^2}}{x}$ if $-1 < x < 0$

(C) $-\tan^{-1} \dfrac{\sqrt{1-x^2}}{x}$ if $0 < x < 1$

(D) $\pi + \tan^{-1} \dfrac{\sqrt{1-x^2}}{x}$ if $-1 < x < 0$

Sol.

EXERCISE – V JEE PROBLEMS

1. The number of real solutions of

$\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2+x+1} = \frac{\pi}{2}$ is **[JEE 99,2]**

(A) zero (B) one (C) two (D) infinite

Sol.

2. Using the principal values, express the following as a single angle $3\tan^{-1}\left(\frac{1}{2}\right) + 2\tan^{-1}\left(\frac{1}{5}\right) + \sin^{-1}\frac{142}{65\sqrt{5}}$ **[REE 99,6]**

Sol.

3. Solve, $\sin^{-1}\frac{ax}{c} + \sin^{-1}\frac{bx}{c} = \sin^{-1}x$,

where $a^2 + b^2 = c^2$, $c \neq 0$. **[REE 2000 (Mains), 3]**

Sol.

4. Solve the equation : $\cos^{-1}(\sqrt{6}x) + \cos^{-1}(3\sqrt{3}x^2) = \frac{\pi}{2}$

[REE 2001 (Mains), 3]

Sol.

5. If $\sin^{-1}\left(x - \frac{x^2}{2} + \frac{x^3}{4} - \ldots\right) + \cos^{-1}\left(x^2 - \frac{x^4}{2} + \frac{x^6}{4} - \ldots\right) = \frac{\pi}{2}$ for 0 < |x| < $\sqrt{2}$ then x equals to

(A) $\frac{1}{2}$ (B) 1 (C) $-\frac{1}{2}$ (D) – 1

[JEE 2001 (Scr.), 1]

Sol.

6. Prove that cos $\tan^{-1}$ sin $\cot^{-1}$ x = $\sqrt{\frac{x^2+1}{x^2+2}}$

[JEE 2002 (Mains) 5]

Sol.

7. Domain of f(x) = $\sqrt{\sin^{-1}(2x) + \frac{\pi}{6}}$ is **[JEE 2003 (Scr.) 3]**

(A) $\left(-\frac{1}{2}, \frac{1}{2}\right]$ (B) $\left[-\frac{1}{4}, \frac{3}{4}\right)$ (C) $\left[-\frac{1}{4}, \frac{1}{4}\right]$ (D) $\left[-\frac{1}{4}, \frac{1}{2}\right]$

Sol.

8. If $\sin(\cot^{-1}(x+1)) = \cos(\tan^{-1} x)$, then x equals

[JEE 2004 (Scr.), 1]

(A) $-\frac{1}{2}$ (B) $\frac{1}{2}$ (C) 0 (D) $\frac{9}{4}$

Sol.

9. Let (x, y) be such that

$\sin^{-1}(ax) + \cos^{-1}(y) + \cos^{-1}(bxy) = \frac{\pi}{2}$ **[JEE 2007, 6]**

Match the statements in **Column I** with statements in **Column II** and indicate your answer by darkening the appropriate bubbles in the 4 × 4 matrix given in the ORS.

Column–I	Column–II
(A) If a = 1 and b = 0, then (x, y)	(P) lies on the circle $x^2 + y^2 = 1$
(B) If a = 1 and b = 1, then (x, y)	(Q) lies on $(x^2 - 1)(y^2 - 1) = 0$
(C) If a = 1 and b = 2, then (x, y)	(R) lies on y = x
(D) If a = 2 and b = 2, then (x, y)	(S) lies on $(4x^2 - 1)(y^2 - 1) = 0$

Sol.

10. If $0 < x < 1$, then

$\sqrt{1+x^2}\ [\{x \cos(\cot^{-1}x) + \sin(\cot^{-1}x)\}^2 - 1]^{1/2}$ equals

(A) $\frac{x}{\sqrt{1+x^2}}$ (B) x (C) $x\sqrt{1+x^2}$ (D) $\sqrt{1+x^2}$ **[JEE 2008,3]**

Sol.

Answer Ex–I SINGLE CORRECT (OBJECTIVE QUESTIONS)

1. D **2.** D **3.** C **4.** D **5.** B **6.** D **7.** A **8.** C
9. B **10.** D **11.** C **12.** A **13.** D **14.** B **15.** B **16.** C
17. A **18.** D **19.** B **20.** B **21.** A **22.** B **23.** C **24.** C
25. A **26.** A **27.** B **28.** C **29.** D **30.** B **31.** B **32.** A
33. C **34.** B **35.** B

Answer Ex–II MULTIPLE CORRECT (OBJECTIVE QUESTIONS)

1. A,B **2.** C,D **3.** B,D **4.** A,B,C **5.** B,C,D **6.** A,C **7.** A,C
8. A,C,D **9.** A,B,C **10.** B,C **11.** B,C **12.** A,C **13.** A,D

Answer Ex–III SUBJECTIVE QUESTIONS

1. 5 **2.** **(i)** $-\sin 1 < x \le 1$, **(ii)** $\cos 2 < x \le 1$, **(iii)** No solution

3. **(i)** $-\frac{\pi}{6}$, **(ii)** $-\frac{\pi}{3}$, **(iii)** $\frac{3\pi}{4}$, **(iv)** $\frac{\pi}{4}$ **4.** **(i)** $5 - 2\pi$, **(ii)** $4\pi - 10$, **(iii)** $2\pi - 6$, **(iv)** $4\pi - 10$, **(v)** $\frac{17\pi}{20}$

5. $\sin^{-1}(\sin\theta) = \begin{cases} \theta - 2\pi, & \frac{3\pi}{2} \le \theta \le \frac{5\pi}{2} \\ 3\pi - \theta, & \frac{5\pi}{2} < \theta \le 3\pi \end{cases}$; $\cos^{-1}(\cos\theta) = \begin{cases} 2\pi - \theta, & \frac{3\pi}{2} \le \theta < 2\pi \\ \theta - 2\pi, & 2\pi \le \theta \le 3\pi \end{cases}$;

$\tan^{-1}(\tan\theta) = \begin{cases} \theta - 2\pi, & \frac{3\pi}{2} < \theta < \frac{5\pi}{2} \\ \theta - 3\pi, & \frac{5\pi}{2} < \theta \le 3\pi \end{cases}$; $\cot^{-1}(\cot\theta) = \begin{cases} \theta - \pi, & \frac{3\pi}{2} \le \theta < 2\pi \\ \theta - 2\pi, & 2\pi < \theta < 3\pi \end{cases}$

6. **(i)** $x = \frac{1}{\sqrt{3}}$, **(ii)** $x = 2$ **7.** $\frac{1+xy}{x-y}$ **9.** 2 **10.** $-\pi$ **11.** $x = 4y^2$

13. **(i)** $\frac{1}{\sqrt{3}}$, **(ii)** 1, **(iii)** $\frac{5\pi}{6}$, **(iv)** $-\frac{\pi}{3}$, **(v)** $\frac{4}{5}$, **(vi)** $\frac{17}{6}$

14. **(i)** $\frac{1}{2}$, **(ii)** -1, **(iii)** $-\frac{\pi}{4}$, **(iv)** $\frac{2\pi}{3}$, **(v)** $\frac{4}{5}$, **(vi)** α

15. **(iv)** $(-\infty, \sec 2) \cup [1, \infty)$

16. **(i)** $-1/3 \le x \le 1$, **(ii)** $\{1, -1\}$, **(iii)** $1 \le x \le 4$, **(iv)** $x \in (-1/2, 1/2)$, $x \ne 0$,
(v) $(3/2, 2]$, **(vi)** $\{7/3, 25/9\}$ **(vii)** $(-2, 2) - \{-1, 0, 1\}$,
(viii) $\{x \mid x = 2n\pi + \frac{\pi}{6}, n \in I\}$ **17.** $\left[\frac{\sqrt{3}}{2}, 1\right]$

19. (i) $x = \frac{1}{2}\sqrt{\frac{3}{7}}$, **(ii)** x = 3, **(iii)** $x = 0, \frac{1}{2}, -\frac{1}{2}$, **(iv)** $x = \frac{3}{\sqrt{10}}$, **(v)** $x = 2 - \sqrt{3}$ or $\sqrt{3}$,

(vi) $x = \frac{1}{2}$, y = 1, **(vii)** $x = \frac{a-b}{1+ab}$

20. 53 **23.** $6\cos^{-1}x - \frac{9\pi}{2}$, so a = 6, $b = -\frac{9}{2}$

24. (i) $\frac{\pi}{2}$, **(ii)** $\frac{\pi}{4}$, **(iii)** arc cot $\left[\frac{2n+5}{n}\right]$, **(iv)** arc tan (x + n) – arc tan x, **(v)** $\frac{\pi}{4}$

25. (i) $x = n^2 - n + 1$ or x = n, **(ii)** x = ab, **(iii)** $x = \frac{4}{3}$

26. (i) $(\cot 2, \infty) \cup (-\infty, \cot 3)$, **(ii)** $\left(\frac{\sqrt{2}}{2}, 1\right]$, **(iii)** $\left(\frac{\sqrt{2}}{2}, 1\right) \cup \left(-1, -\frac{\sqrt{2}}{2}\right)$

Answer Ex–IV ADVANCED SUBJECTIVE QUESTIONS

3. (a) $\text{arc}\cot\left[\frac{2n+5}{n}\right]$, **(b)** $\frac{\pi}{4}$, **(c)** arc tan (x + n) – arc tan x, **(d)** $\frac{\pi}{4}$, **(e)** $\frac{\pi}{2}$

4. $(\alpha^2+\beta^2)(\alpha+\beta)$ **5.** K = 2, $\cos\frac{\pi^2}{4}, 1$ & $\cos\frac{\pi^2}{4}, -1$ **6.** x = 1, y = 2, & x = 2 ; y = 7

7. $X = Y = \sqrt{3-a^2}$ **8.** (A)–P,Q,R,S ; (B)–P,Q ; (C)–P,R,S ; (D)–P,R,S

10. (a) $(\cot 2, \infty) \cup (-\infty, \cot 3)$, **(b)** $\left(\frac{\sqrt{2}}{2}, 1\right]$, **(c)** $\left(\frac{\sqrt{2}}{2}, 1\right) \cup \left(-1, -\frac{\sqrt{2}}{2}\right)$ **11.** $\left(\tan\frac{1}{2}, \cot 1\right]$

12. 3388	**13.** k = 25	**14.** $\frac{3\pi}{4}$	**15.** $x \in (-1, 1)$	**16.** $a \in [-2\pi, \pi] - \{0\}$	
17. A	**18.** C	**19.** B	**20.** D	**21.** C	**22.** D

Answer Ex–V JEE PROBLEMS

1. C	**2.** π	**3.** $x \in \{-1, 0, 1\}$	**4.** x = 1/3	**5.** B
7. D	**8.** A	**9.** (A)→P ; (B)→Q ; (C)→P ; (D)→S		**10.** C